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Hyperquaternion Conformal Groups

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Hyperquaternion Conformal Groups

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Abstract. The paper gives a new representation of conformal groups in n dimensions in terms of hyperquaternions defined as tensor products of quaternion algebras (or a subalgebra thereof). Being Clifford algebras, hyperquaternions provide a good representation of pseudo-orthogonal groups such as $O(p+1, q+1)$ isomorphic to the nD conformal group with $n = p + q$. The representation yields simple expressions of the generators, independently of matrices or operators. The canonical decomposition and the invariants are discussed. As application, the $4D$ relativistic conformal group is detailed together with a worked example. Finally, the formalism is compared to the operator representation. Potential uses include in particular, conformal geometry, computer graphics and conformal field theory.

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1. Introduction

The $4D$ conformal group which contains the Poincaré group, is the group of transformations $x \to x'$ of the Minkowski space $E_{1,3}$ (with signature $+---$) such that $ds'^2 = \lambda^2 ds^2$ where λ is a real number and $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ the metric. This group plays a major role in physics in particular in electromagnetism, general relativity and conformal field theories [4, 18]. Maxwell's equations (in vacuum) are covariant with respect to the $4D$ conformal group and in general relativity, twistors constitute an algebraic representation of it [21, 25]. Generalizing to the nD case (with an arbitrary signature), we shall call them conformal groups $\text{Conf}_n(p, q)$ with $n = p + q$. Due to the isomorphism $\text{Conf}_n(p, q) \simeq O(p+1, q+1)$, the conformal groups can be expressed as orthogonal groups and represented in particular by Clifford algebras $C_{n+2}(p + 1, q + 1)$ [2]. Various representations (in specific dimensions or signatures) exist and are often formulated in terms of matrices [15, 16, 22, 24]. Though we shall focus on the algebraic properties of the conformal groups, geometric aspects have been developed in particular by Hestenes, the Lasenbys, Dorst, Dechant and Hitzer. Adopting the general method presented in [2], we shall introduce a new algebraic representation in terms of hyperquaternions. Historically, hyperquaternions are rooted in the works of Clifford, Lipschitz and Moore [6, 17, 19, 20]. Clifford, applying Grassmann's ideas, introduced his algebras as a tensor product of quaternions and gave a proof thereof. Lipschitz established the formula of nD Euclidean rotations and thereby rediscovered, independently of Clifford, the (even) Clifford algebra (composed of products of an even number of generators). Moore was to call Lipschitz's algebras "hyperquaternions" and gave a canonical decomposition of Euclidean rotations which has been extended to pseudo-Euclidean rotations by the authors [9]. In recent papers, we have applied hyperquaternions to express the unitary, unitary symplectic groups and the Poincaré groups with dual hyperquaternions [7, 9, 10]. Though dual hyperquaternions provide a correct representation of the Poincaré groups, the introduction of a dual element (squaring to zero) might appear as somewhat unnatural. Here, by adding two dimensions to the initial space, we shall introduce such elements within the hyperquaternion framework thereby extending the Poincaré groups to the nD conformal groups. Concerning the isomorphism of hyperquaternions with standard Clifford algebras, a modern proof has been given in [10]. Hyperquaternions yield an intrinsically defined tensor product (without generators), a unique multivector structure (after a generator choice) as well as simple expressions of the generators. The result is an efficient calculus with a straightforward implementation. Hyperquaternions might also open new perspectives of unification. After a few preliminaries, we present the general framework of nD conformal groups. Then, we discuss the canonical decomposition and the invariants. As example, the $4D$ conformal group is examined and a worked example is provided. Finally, the hyperquaternionic approach is compared to an operator representation. The Lie algebra of the nD conformal group and the explicit multivector structure of the $4D$ case are given respectively in Appendix A and B. Potential applications include in particular, conformal geometry, computer graphics and conformal field theory.

2. Quaternions, Hyperquaternions and Multivectors

Quaternions, constituting the quaternion algebra ($\mathbb{H}$), are a set of four real numbers [7, 9, 10, 12, 13, 23]

$$a = a_1 + a_2 i + a_3 j + a_4 k \tag{2.1}$$

where the elements i, j, k satisfy the fundamental formula

$$i^2 = j^2 = k^2 = ijk = -1. \tag{2.2}$$

The quaternion conjugate is $a_c = a_1 - a_2 i - a_3 j - a_4 k$ with $(ab)_c = b_c a_c$.

Hyperquaternions are defined as tensor products of quaternions $\mathbb{H}$, $\mathbb{H}\otimes\mathbb{H}$, $\mathbb{H}\otimes\mathbb{H}\otimes\cdots\otimes\mathbb{H}$ and their subalgebras $\mathbb{C}$, $\mathbb{H}\otimes\mathbb{C}$, $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{C}$, *etc.*. The tensor product is uniquely defined via formulas such as

$$(i,j,k)\otimes 1\otimes 1=(i,j,k)\,, \tag{2.3}$$

$$1\otimes(i,j,k)\otimes 1=(I,J,K)\,, \tag{2.4}$$

$$1\otimes 1\otimes(i,j,k)=(l,m,n) \tag{2.5}$$

where $(i,j,k),(I,J,K),(l,m,n)$ are distinct commuting quaternionic systems. We shall use the algebra $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$ in the $4D$ example. Hyperquaternions yield all real, complex and quaternionic square matrices via the isomorphism $\mathbb{H}\otimes\mathbb{H}\simeq m(4,\mathbb{R})$ where $m(4,\mathbb{R})$ stands for real square matrices of degree 4.

Hyperquaternions constitute Clifford algebras $C_n(p,q)$, over real numbers, having $n=p+q$ generators e_i such that $e_ie_j+e_je_i=0$ $(i\neq j)$, $e_i^2=+1$ (p generators) and $e_i^2=-1$ (q generators) where the generators have a simple compact form (for example $e_1=kJ$, $e_2=kKl$, *etc.*). Products of generators yield multivectors (V_k) ($0\leq k\leq n$), such as scalars (V_0), vectors e_i (V_1), bivectors e_ie_j $(i<j)$ (V_2), trivectors $e_ie_je_k$ $(i<j<k)$ (V_3) etc.. The products of an even number of e_i constitute the subalgebra C^+, the rest of the algebra is C^-. The commutator of two general elements A,B is

$$[A,B]=(AB-BA)/2. \tag{2.6}$$

The conjugate A_c of a general element A is obtained by replacing the e_i by their opposite $-e_i$ and reversing the order of the elements

$$(A_c)_c=A,\quad (AB)_c=(B_c)(A_c)\,. \tag{2.7}$$

The interior and exterior products of two vectors a,b are given by

$$2a\cdot b=(ab+ba)\,,\quad 2a\wedge b=(ab-ba)\,. \tag{2.8}$$

More generally, for products between a vector and a multivector $A_p=a_1\wedge a_2\wedge\cdots\wedge a_p$ $(2\leq p<n)$, one has

$$2a\cdot A_p = \left[aA_p-(-1)^p A_pa\right], \tag{2.9}$$

$$2a\wedge A_p = \left[aA_p+(-1)^p A_pa\right]. \tag{2.10}$$

Products between multivectors are defined by [5]

$$A_p\wedge B_q = a_1\wedge(a_2\wedge\ldots\wedge a_p\wedge B_q), \tag{2.11}$$

$$\begin{aligned} A_p\cdot B_q &= (a_1\wedge\ldots\wedge a_{p-1})\cdot(a_p\cdot B_q)\\ &= (-1)^{p(q+1)}B_q\cdot A_p \qquad (p\leq q). \end{aligned} \tag{2.12}$$

For bivectors one has, in particular

$$B_1B_2=B_1\cdot B_2+B_1\wedge B_2+[B_1,B_2] \tag{2.13}$$

with

$$B_1\wedge B_2=V_4\left[\frac{1}{2}(B_1B_2+B_2B_1)\right]. \tag{2.14}$$

3. Conformal Groups in n Dimensions

We develop here the hyperquaternion conformal groups in n dimensions. The method consists in embedding the nD space in an $(n+2)\,D$ affine space according to the method presented in [2]. We start with the general framework, apply it to the restricted conformal group and then discuss the canonical decomposition as an extension of Moore's method and the invariants [9].

3.1. General Framework

To express the conformal group, consider the real hyperquaternion algebra $C_n(p,q)$ having $n=p+q$ generators e_i, a vector space $(E_n,\, x=\sum_{i=1}^{n} x_i e_i)$ and the metric

$$ds^2 = dx^2 = \left(dx_1^2 + \cdots + dx_p^2\right) - \left(dx_{p+1}^2 + \cdots + dx_{p+q}^2\right). \tag{3.1}$$

Vectors are said to be timelike, spacelike and isotropic if respectively x^2 is positive, negative or nil. The conformal group of E_n is defined as the group of transformations $x \to x'$ such that

$$ds'^2 = \lambda(x)^2 ds^2 \quad (\lambda(x) \in \mathbb{R}). \tag{3.2}$$

To obtain the latter, one embeds the algebra $C_n(p,q)$ in the algebra $C_{n+2}(p+1,q+1)$ with two additional generators e_0, e_{n+1} $\left(e_0^2 = 1, e_{n+1}^2 = -1\right)$. An affine space is introduced by

$$\begin{aligned} X &= \left(\frac{x^2-1}{2}\right) e_0 + x + \left(\frac{x^2+1}{2}\right) e_{n+1} \\ &= x^2\varepsilon_1 + x + \varepsilon_2 \end{aligned} \tag{3.3}$$

with $X^2 = 0$ and

$$\varepsilon_1 = \frac{e_0 + e_{n+1}}{2}, \quad \varepsilon_2 = \frac{e_{n+1} - e_0}{2}, \quad \varepsilon_1 \wedge \varepsilon_2 = \frac{e_0 e_{n+1}}{2}, \quad \varepsilon_1^2 = \varepsilon_2^2 = 0. \tag{3.4}$$

The nD conformal group of E_n is isomorphic to the orthogonal group $O(p+1,q+1)$ [2]. The latter can be constructed from orthogonal symmetries with respect to a plane (going through the origin), perpendicular to a unit vector u (time or spacelike) and given by

$$X' = -\frac{uXu}{u^2}. \tag{3.5}$$

By combining these symmetries, one obtains the formulas

$$X' = \pm a X a_c \quad (a a_c = \pm 1) \tag{3.6}$$

where $a \in C_{n+2}^{+}(p+1,q+1)$ or $C_{n+2}^{-}(p+1,q+1)$. The nD conformal group contains the Poincaré group, transversions (also called special conformal transformations) and dilations. The number of parameters equals $\frac{(n+2)(n+1)}{2}$; its Lie algebra is given in Appendix A.

If p and q are odd, the conformal group, like the Poincaré group, has four connected components. In all other cases, it has only two connected components [2, p. 88]. An important physical example of the first type is the

Table 1. Hyperquaternion conformal group (R: rotation, T: translation, V: transversion, D: dilation; $m(x) = (1 + Kx)(1 + xK)$)

	R	T	V	D
a	$e^{e_i e_j \frac{m^{ij}}{2}}$	$e^{\varepsilon_1 P}$	$e^{\varepsilon_2 K}$	$e^{e_0 e_{n+1} \frac{\varphi}{2}}$
$X' (= aXa_c)$	$x^2\varepsilon_1$ $+x' + \varepsilon_2$	$x^2\varepsilon_1$ $+x' + \varepsilon_2$	$x^2\varepsilon_1 + x'$ $+\varepsilon_2 m$	$x^2\varepsilon_1 e^{-\varphi}$ $+x' + \varepsilon_2 e^{\varphi}$
x'	axa_c	$(x + P)$	$x + Kx^2$	x
$\alpha(x)$	1	1	$m(x)$	e^{φ}
$y(x) = \frac{x'}{\alpha(x)}$	axa_c	$(x + P)$	$\frac{x+Kx^2}{m(x)}$	$xe^{-\varphi}$

Table 2. Multiplication rules

	ε_1	ε_2	$e_0 e_{n+1}$
ε_1	0	$\frac{(e_0 e_{n+1} - 1)}{2}$	ε_1
ε_2	$-\frac{(e_0 e_{n+1} + 1)}{2}$	0	$-\varepsilon_2$
$e_0 e_{n+1}$	$-\varepsilon_1$	ε_2	1

relativistic space-time ($p = 1, q = 3$), which we shall develop below. Since all other connected components can be derived from the restricted conformal group $\mathrm{Conf}_+^{\uparrow}$ (orthochronous and of determinant 1) we shall now focus on the latter below.

3.2. Restricted Conformal Group

The restricted conformal group is defined as

$$X' = aXa_c = X'^0 e_0 + x' + X'^{n+1} e_{n+1} \tag{3.7}$$

with $aa_c = 1, a \in C^+_{n+2}(p + 1, q + 1)$. Writing $X' = \alpha(x)Y$ with

$$Y = \left(\frac{y^2 - 1}{2}\right) e_0 + y(x) + \left(\frac{y^2 + 1}{2}\right) e_{n+1} \tag{3.8}$$

where $Y^2 = 0$, one finds $\alpha(x) = X'^{n+1} - X'^0$ and obtains the expression of the conformal transform of x [2]

$$y(x) = \frac{x'}{\alpha(x)} = \frac{x'}{X'^{n+1} - X'^0}. \tag{3.9}$$

Explicitly, the conformal group is constituted by (pseudo-Euclidean) rotations, translations, transversions and dilations as indicated in Table 1 where X' is obtained via the multiplication rules given in Table 2.

To verify that the above transformations are indeed conformal ones, one might proceed as follows. Assuming a to be constant (independent of x), one has

$$dX' = adXa_c = d\alpha(x)Y + \alpha(x)dY. \tag{3.10}$$

Hence,

$$dX'^2 = dX^2 = dx^2 = ds^2 = \alpha(x)^2 dY^2 = \alpha(x)^2 dy^2 \tag{3.11}$$

where we have used $Y^2 = 0$ (and thus $YdY + dYY = 0$). Finally, one finds

$$ds^{'2} \equiv dy^2 = \frac{ds^2}{\alpha(x)^2} = \lambda(x)^2 ds^2 \tag{3.12}$$

which concludes the verification. The conservation of angles results from

$$\left[d\left(u' + v'\right)\right]^2 = \lambda(x)^2 \left[d\left(u + v\right)\right]^2 \tag{3.13}$$

yielding

$$du'.dv' = \lambda(x)^2 du.dv. \tag{3.14}$$

A standard simple form of nD transversions $x \to y(x)$ is derived within the Clifford algebra $C_n(p,q)$ from

$$\begin{aligned} y(x) &= \frac{x + Kx^2}{(1 + Kx)(1 + xK)} = \frac{x^2\left(\frac{x}{x^2} + K\right)}{\left(\frac{x}{x^2} + K\right) xx \left(\frac{x}{x^2} + K\right)} \\ &= \frac{x^{-1} + K}{\left(x^{-1} + K\right)^2} = \left(x^{-1} + K\right)^{-1} \end{aligned} \tag{3.15}$$

(where x^2 is a scalar commuting with K) and giving

$$[y(x)]^{-1} = x^{-1} + K. \tag{3.16}$$

3.3. Canonical Decomposition

Here, we shall apply the canonical decomposition of pseudo-Euclidean rotations given in [9].

Considering the restricted group $SO^+(p+1, q+1)$ with $n = p + q = 2k$ (or $2k+1$), a rotation within that group can be decomposed into a maximum of $k + 1$ orthogonal commuting simple planes B_i [9]

$$a = e^{\frac{\Phi_1}{2}B_1} e^{\frac{\Phi_2}{2}B_2} \cdots e^{\frac{\Phi_m}{2}B_m} \tag{3.17}$$

where a simple plane is defined by $B_i \wedge B_i = 0$ entailing that B_i^2 is a scalar, $B_i^2 \in \{\pm 1, 0\}$. The simple planes B_i can be expressed as

$$B_i = M_i + \varepsilon_1 \wedge P_i + \varepsilon_2 \wedge K_i + \varepsilon_1 \wedge \varepsilon_2 D_i \tag{3.18}$$

with $M_i \in C^+_{n+2}(p+1, q+1)$, and $(P_i, K_i \in E_n,\ D_i \in \mathbb{R})$. The bivector M_i represents the rotation, the vectors P_i, K_i respectively the translation and transversion, D_i the dilation. Expanding the relation $B_i \wedge B_i = 0$, one obtains

$$\begin{aligned} 0 &= M_i \wedge M_i + 2\varepsilon_1 \wedge (M_i \wedge P_i) \\ &\quad +2\varepsilon_2 \wedge (M_i \wedge K_i) + 2\varepsilon_1 \wedge \varepsilon_2 (D_i M_i - P_i \wedge K_i) \end{aligned} \tag{3.19}$$

and thus

$$M_i \wedge M_i = 0, \quad D_i M_i = P_i \wedge K_i, \tag{3.20}$$

$$M_i \wedge P_i = M_i \wedge K_i = 0. \tag{3.21}$$

Hence, M_i is a simple plane and the vectors P_i, K_i are coplanar with M_i and anticommute with it ($M_i P_i = -P_i M_i$), and similarly for K_i. Developing the term B_i^2 yields

$$B_i^2 = M_i^2 + \frac{D_i^2}{4} + P_i \cdot K_i \in \{\pm 1, 0\} \tag{3.22}$$

where $P_i \cdot K_i = \frac{1}{2}(P_i K_i + K_i.P_i)$ is the interior product of the two vectors. For $B_i^2 \in \{-1, 1, 0\}$ one has respectively

$$e^{\frac{\Phi_i}{2}B_i} = \cos\frac{\Phi_i}{2} + B_i \sin\frac{\Phi_i}{2}, \tag{3.23}$$

$$e^{\frac{\Phi_i}{2}B_i} = \cosh\frac{\Phi_i}{2} + B_i \sinh\frac{\Phi_i}{2}, \tag{3.24}$$

$$e^{\frac{\Phi_i}{2}B_i} = 1 + B_i\frac{\Phi_i}{2}. \tag{3.25}$$

Each component $f_i = e^{\frac{\Phi_i}{2}B_i}$ can be decomposed via the Liouville theorem and its extension to pseudo-Euclidean spaces by Haantjes [14] into a rotation (R_i), a translation (T_i), a transversion (V_i) and a dilation (D_i) for example as

$$f_i = R_i T_i V_i D_i \tag{3.26}$$

leading for the entire transformation $f = f_1 f_2 \cdots f_m$ to

$$\begin{aligned} f &= (R_1T_1V_1D_1)(R_2T_2V_2D_2)\cdots(R_mT_mV_mD_m) \\ &= (R_1R_2\cdots R_m)(T_1T_2\cdots T_m)(V_1V_2\cdots V_m)(D_1D_2\cdots D_m) \\ &= RTVD \end{aligned} \tag{3.27}$$

where we have used the commutativity of distinct orthogonal simple planes. Hence, one obtains a complete decomposition.

3.4. Invariants of the Restricted Conformal Group

The conformal group being isomorphic to $O(p+1, q+1)$, the invariants are those of the latter group. For $SO(2m)$ or $SO(2m+1)$, one has a maximum of m independent Casimir operators commuting with the bivector B generating the group [3]. For the restricted conformal group $SO^+(p+1, q+1)$, the invariants can be obtained via the canonical decomposition presented in [9]. Writing

$$B = b_1 B_1 + b_2 B_2 + \cdots + b_m B_m \tag{3.28}$$

where B_m are orthogonal commuting simple planes and b_i are scalars, one computes the bivectors

$$P_1 = B, \tag{3.29}$$

$$P_2 = (B \wedge B) \cdot B, \tag{3.30}$$

$$P_m = \underbrace{(B \wedge B \wedge \cdots \wedge B)}_{m \text{ factors}} \cdot \underbrace{(B \wedge B \wedge \cdots \wedge B)}_{m-1 \text{ factors}} \tag{3.31}$$

and the scalars

$$S_1 = P_1 \cdot P_1,\ S_2 = P_2 \cdot P_1,\ S_m = P_m \cdot P_1 \tag{3.32}$$

which commute with B and thus constitute m invariants.

4. Example: $4D$ Conformal Group

The $4D$ conformal group is of great importance in physics [4, 18]. We first introduce the algebra, followed by a worked example with a canonical decomposition and the construction of the invariants. Finally, the hyperquaternion representation is compared to an operator representation.

4.1. Algebra

Consider the hyperquaternion algebra $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$ $[\simeq C_6\,(2,4)]$ having six generators

$$e_0 = kI, \quad e_1 = kJ, \quad e_2 = kKl, \quad e_3 = kKm, \quad e_4 = kKn, \quad e_5 = j \tag{4.1}$$

and the multivector structure given in Appendix A. The $4D$ vector space $E_4 = \text{span}(e_1, \ldots, e_4)$ has the metric (with $x = \sum_{i=1}^{4} x_i e_i$)

$$ds^2 = dx^2 = dx_1^2 - dx_2^2 - dx_3^2 - dx_4^2. \tag{4.2}$$

The restricted conformal group is expressed by

$$X' = aXa_c \ \left(aa_c = 1, \quad a \in C_6^+(2,4)\right) \tag{4.3}$$

with $X = X^0 e_0 + x + X^5 e_5$ (and similarly for X'). The group contains the following transformations:

- spatial rotations

$$a = e^{B\frac{\theta}{2}} \quad \left[B^2 = -1, \quad B \in (l, m, n)\right] \tag{4.4}$$

- boosts

$$a = e^{B\frac{\theta}{2}} \quad \left[B^2 = 1, \quad B \in (Il, Im, In)\right] \tag{4.5}$$

- space-time translations

$$a = e^{\varepsilon_1 P} = 1 + \varepsilon_1 P \quad \left[P \in E_4, \quad \varepsilon_1 = \frac{kI + j}{2}, \quad \varepsilon_1^2 = 0\right] \tag{4.6}$$

- transversions

$$a = e^{\varepsilon_2 K} = 1 + \varepsilon_2 K \quad \left[K \in E_4, \quad \varepsilon_2 = \frac{j - kI}{2}, \quad \varepsilon_2^2 = 0\right] \tag{4.7}$$

- dilations

$$a = e^{-iI\frac{\varphi}{2}} \quad (\varphi \in \mathbb{R}) \tag{4.8}$$

with a total of 15 parameters. A combination of the above transformations leads to the element f

$$X' = fXf_c \quad \left(ff_c = 1, \quad f \in C_6^+(2,4)\right). \tag{4.9}$$

The canonical decomposition of f yields at most three orthogonal commuting simple planes B_i

$$f = e^{B_1\frac{\Phi_1}{2}} e^{B_2\frac{\Phi_2}{2}} e^{B_3\frac{\Phi_3}{2}}, \quad B_i^2 \in \{\pm 1, 0\} \tag{4.10}$$

with $B = b_1 B_1 + b_2 B_2 + b_3 B_3$ the bivector of f and

$$B_i = M_i + \varepsilon_1 P_i + \varepsilon_2 K_i + \varepsilon_1 \wedge \varepsilon_2 D_i. \tag{4.11}$$

One thus obtains a maximum number of three invariants.

4.2. Worked Example

As worked example, consider a transformation $X' = aXa_c$ generated by a dilation, followed by a transversion and a rotation such as

$$\begin{aligned} a &= e^{n\frac{\theta}{2}} e^{\varepsilon_2 (kKm)} e^{e_0 e_{n+1} \frac{\varphi}{2}} && (4.12) \\ &= \sqrt{\frac{2}{3}}(1+n) + \frac{1}{2\sqrt{6}} \left[(J - iK)(l - m) \right] - \frac{1}{\sqrt{6}} iI (1+n) && (4.13) \end{aligned}$$

with $\left(\tan\frac{\theta}{2} = 1, \tanh\frac{\varphi}{2} = \frac{1}{2}\right)$ and the bivector B generating the transformation

$$B = n - \frac{1}{2} iI + \frac{1}{4}(J - iK)(l - m). \tag{4.14}$$

Applying the canonical decomposition presented in [9], one finds

$$a = e^{B_1 \frac{\Phi_1}{2}} e^{B_2 \frac{\Phi_2}{2}} \tag{4.15}$$

where B_1, B_2 are two commuting orthogonal simple planes

$$\begin{aligned} B_1 &= \frac{1}{10}(J - iK)(3l - m), && (4.16) \\ B_2 &= -\frac{1}{10}(J - iK)(l + 3m) && (4.17) \end{aligned}$$

with $B_1^2 = -1, B_2^2 = 1 \left(\tan\frac{\Phi_1}{2} = 1, \tanh\frac{\Phi_2}{2} = \frac{1}{2}\right)$. To obtain the invariants, one computes, as indicated above, the quantities

$$\begin{aligned} P_1 &= B, && (4.18) \\ P_2 &= (B \wedge B) \cdot B = \frac{1}{2} + iI + \frac{1}{4}(J - iK)(l + m), && (4.19) \\ P_3 &= (B \wedge B \wedge B) \cdot (B \wedge B) = 0 && (4.20) \end{aligned}$$

and the scalars

$$S_1 = P_1 \cdot P_1 = -\frac{3}{4}, \quad S_2 = P_2 \cdot P_1 = -1, \quad S_3 = P_3 \cdot P_1 = 0 \tag{4.21}$$

yielding the two conformal invariants S_1, S_2.

4.3. Other Representation

Above, we have given an algebraic representation of the nD conformal groups. Other representations exist in particular in terms of operators with the correspondence [2, p. 135]

- rotations: $e_i e_j \to x_i \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial x_i}$,
- translations: $(e_0 + e_{n+1})\, e_i \to \frac{\partial}{\partial x_i}$,
- transversions: $(e_{n+1} - e_0)\, e_i \to x^2 \frac{\partial}{\partial x_i} - 2x_i x^k \frac{\partial}{\partial x_k}$,
- dilations: $e_0 e_{n+1} \to x_i \frac{\partial}{\partial x_i}$.

Both representations lead to the same Lie algebra, given in Appendix A. Operators are used in quantum mechanics and quantum field theories [1]. Physical applications of conformal groups include in particular electromagnetism, general relativity and conformal field theory [4,18].

5. Conclusion

In this paper, we have developed a new algebraic representation of conformal groups in n dimensions in terms of hyperquaternions. The representation is distinct from matrix ones and gives simple expressions of the generators. After the general formalism, the canonical decomposition and the invariants have been discussed. As concrete example, the $4D$ relativistic case has been detailed together with a worked example. Finally, the representation has been compared to an operator representation. It is hoped that the hyperquaternionic approach might advance the understanding of the algebraic structure of conformal groups, provide an efficient operational calculus and open new unification perspectives. Potential applications include in particular, conformal geometry, computer graphics and conformal field theory.

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Appendix A. Lie Algebra of the nD-Conformal Group

Consider an nD space embedded in an $n+2$ hyperquaternion algebra with the generators $e_0, e_1, \ldots, e_n, e_{n+1}$. The Lie generators of the rotations, translations, transversions and dilations of the restricted conformal group are respectively

$$M_{ij} = \frac{1}{2} e_i e_j \quad (1 \leq i, j \leq n, i \neq j), \tag{A.1}$$

$$P_i = \left(\frac{e_0 + e_{n+1}}{2} \right) e_i, \qquad K_i = \left(\frac{e_{n+1} - e_0}{2} \right) e_i, \tag{A.2}$$

$$D = \frac{e_0 e_{n+1}}{2}. \tag{A.3}$$

One derives easily the following Lie commutators $[A,B]=AB-BA$, with $\eta_{ij}=(e_ie_j+e_je_i)/2$.

$$[M_{ij},M_{kl}] = \eta_{jk}\ M_{il}+\eta_{il}\ M_{jk}-\eta_{jl}\ M_{ik}-\eta_{ik}\ M_{jl}, \quad (A.4)$$
$$[M_{ij},P_k] = \eta_{jk}P_i-\eta_{ik}\ P_j, \quad [M_{ij},K_k]=\eta_{jk}K_i-\eta_{ik}\ K_j, \quad (A.5)$$
$$[M_{ij},D] = [P_i,P_j]=[K_i,K_j]=0, \quad (A.6)$$
$$[K_i,P_j] = 2\eta_{ij}D+M_{ij}, \quad [D,P_i]=-P_i,\ \ [D,K_i]=K_i. \quad (A.7)$$

Appendix B. Multivector structure of $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$

A general hyperquaternion A is a set of 64 terms which can be grouped into a set of 16 quaternions $[q_i]=a_i+b_il+c_im+d_in$ with respect to the sets $ijk/IJK/lmn$

$$\begin{aligned} A = & [q_1]+I[q_2]+J[q_3]+K[q_4] \\ & +i[q_5]+iI[q_6]+iJ[q_7]+iK[q_8] \\ & +j[q_9]+jI[q_{10}]+jJ[q_{11}]+jK[q_{12}] \\ & +k[q_{13}]+kI[q_{14}]+kJ[q_{15}]+kK[q_{16}] \end{aligned} \quad (B.1)$$

yielding a multiplication table which can be implemented (algebraically or numerically) on `Mathematica` using its quaternion product $[q_i]**[q_j]$. The complete multivector structure is given below, where $e_{0123}=e_0e_1e_2e_3$, etc..

$$\begin{bmatrix}
1 & l=e_{34} & m=e_{42} & n=e_{23} \\
I=e_{1234} & I\ l=e_{21} & I\ m=e_{31} & I\ n=e_{41} \\
J=e_{2034} & J\ l=e_{02} & J\ m=e_{03} & J\ n=e_{04} \\
K=e_{10} & Kl=e_{1034} & Km=e_{0124} & Kn=e_{1023} \\
i=e_{012345} & il=e_{1025} & im=e_{1035} & in=e_{1045} \\
iI=e_{50} & iI\ l=e_{3045} & iI\ m=e_{4025} & iI\ n=e_{2035} \\
iJ=e_{51} & iJ\ l=e_{3145} & iJ\ m=e_{4125} & iJ\ n=e_{2135} \\
iK=e_{2345} & iKl=e_{52} & iKm=e_{53} & iKn=e_{54} \\
j=e_5 & jl=e_{345} & jm=e_{542} & jn=e_{235} \\
jI=e_{12345} & jI\ l=e_{215} & jI\ m=e_{531} & jI\ n=e_{541} \\
jJ=e_{32450} & jJ\ l=e_{025} & jJ\ m=e_{035} & jJ\ n=e_{045} \\
jK=e_{105} & jKl=e_{10345} & jKm=e_{01245} & jKn=e_{10235} \\
k=e_{10234} & kl=e_{012} & km=e_{013} & kn=e_{014} \\
kI=e_0 & kI\ l=e_{034} & kI\ m=e_{204} & kI\ n=e_{023} \\
kJ=e_1 & kJ\ l=e_{134} & kJ\ m=e_{214} & kJ\ n=e_{123} \\
kK=e_{324} & kKl=e_2 & kKm=e_3 & kKn=e_4
\end{bmatrix} \quad (B.2)$$

A `Mathematica` notebook, concerning $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{C}$ is provided in [8].

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