Appendix of
Region growing scheme as local optimization of discrete region-based energies for multi-region segmentation

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This is the appendix of the work titled *"Local optimization of discrete region-based energies for multi-region segmentation"*. Here we develop the mathematical derivations that lead to the variations of the data terms presented in the above work, as well as the regularization term. For each derived term, the energy variation caused by label modification at pixel $\boldsymbol{v}$ is studied. We denote by $i = \phi_{\boldsymbol{v}}$ and m the initial and candidate labels, respectively. With a view to conciseness, the tested labeling $\phi_{\boldsymbol{v},m}$ is shortened to ϕ', whereas variation of energy $J[\phi'] - J[\phi]$ is shortened to $\Delta_{\boldsymbol{v}} J[\phi]$.

1 Global uniform modeling

We detail the variation of the global uniform modeling term addressed in section 3.1 "Global uniform modeling" and based on the following penalty function:

$$f[l, \boldsymbol{p}, \phi] = \|\boldsymbol{I_p} - \boldsymbol{\mu}_l[\phi]\|^2$$

where $\boldsymbol{\mu}_l[\phi]$ is the average color over region $\mathcal{R}_l$:

$$\boldsymbol{\mu}_l[\phi] = \frac{1}{|\mathcal{R}_l|} \sum_{\boldsymbol{q} \in \mathcal{D}} \delta(\phi_{\boldsymbol{q}}, l)\, \boldsymbol{I_q}$$

It is clear that label modification only affects data terms over $\mathcal{R}_i$ and $\mathcal{R}_m$, hence we have:

$$\begin{gathered}\Delta_{\boldsymbol{v}} J[\phi] = \sum_{\boldsymbol{p} \in \mathcal{D}} \Big\{ \delta(\phi'_{\boldsymbol{p}}, m)\, \|\boldsymbol{I_p} - \boldsymbol{\mu}_m[\phi']\|^2 \\ -\delta(\phi_{\boldsymbol{p}}, m)\, \|\boldsymbol{I_p} - \boldsymbol{\mu}_m[\phi]\|^2 + \delta(\phi'_{\boldsymbol{p}}, i)\, \|\boldsymbol{I_p} - \boldsymbol{\mu}_i[\phi']\|^2 \\ -\delta(\phi_{\boldsymbol{p}}, i)\, \|\boldsymbol{I_p} - \boldsymbol{\mu}_i[\phi]\|^2 \Big\}\end{gathered}$$

where candidate means $\boldsymbol{\mu}_i[\phi']$ and $\boldsymbol{\mu}_m[\phi']$ are:

$$\boldsymbol{\mu}_l[\phi'] = \begin{cases} \dfrac{\boldsymbol{\mu}_l[\phi]\,|\mathcal{R}_l| - \boldsymbol{I_v}}{|\mathcal{R}_l| - 1} & \text{if } l = i \\ \dfrac{\boldsymbol{\mu}_l[\phi]\,|\mathcal{R}_l| + \boldsymbol{I_v}}{|\mathcal{R}_l| + 1} & \text{if } l = m \\ \boldsymbol{\mu}_l[\phi] & \text{otherwise} \end{cases} \tag{1}$$

It follows that energy variation is made up of two distinct parts, i.e. the swap of deviations due to $\boldsymbol{v}$ itself and the deviations of all unchanged pixels belonging to regions $\mathcal{R}_i$ and $\mathcal{R}_m$:

$$\begin{gathered}\Delta_{\boldsymbol{v}} J[\phi] = \|\boldsymbol{I_v} - \boldsymbol{\mu}_m[\phi']\|^2 - \|\boldsymbol{I_v} - \boldsymbol{\mu}_i[\phi]\|^2 \\ + \sum_{\boldsymbol{p} \in \mathcal{D} \setminus \{\boldsymbol{v}\}} \Big\{ \delta(\phi_{\boldsymbol{p}}, m) \left(\|\boldsymbol{I_p} - \boldsymbol{\mu}_m[\phi']\|^2 - \|\boldsymbol{I_p} - \boldsymbol{\mu}_m[\phi]\|^2 \right) \\ +\delta(\phi_{\boldsymbol{p}}, i) \left(\|\boldsymbol{I_p} - \boldsymbol{\mu}_i[\phi']\|^2 - \|\boldsymbol{I_p} - \boldsymbol{\mu}_i[\phi]\|^2 \right) \Big\}\end{gathered}$$

Expanding squared ℓ^2 distances with rule

$$\|\boldsymbol{a} - \boldsymbol{b}\|^2 = \|\boldsymbol{a}\|^2 + \|\boldsymbol{b}\|^2 - 2\boldsymbol{a} \cdot \boldsymbol{b},$$

the variation simplifies to:

$$\Delta_{\boldsymbol{v}} J\left[\phi\right] = \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\mu}_m[\phi']\right\|^2 - \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\mu}_i[\phi]\right\|^2$$
$$+|\mathcal{R}_m| \left\|\boldsymbol{\mu}_m[\phi'] - \boldsymbol{\mu}_m[\phi]\right\|^2 - (|\mathcal{R}_i| - 1) \left\|\boldsymbol{\mu}_i[\phi'] - \boldsymbol{\mu}_i[\phi]\right\|^2$$

2 Local uniform modeling

We detail the variation of the local uniform modeling term addressed in section 3.2 ("Local uniform modeling") and based on the following penalty function:

$$f\left[l, \boldsymbol{p}, \phi\right] = \left\|\boldsymbol{I}_{\boldsymbol{p}} - \boldsymbol{\nu}_l[\boldsymbol{p}, \phi]\right\|^2$$

where $\boldsymbol{\nu}_l[\boldsymbol{p}, \phi]$ is the normally-weighted mean of colors in region $\mathcal{R}_l$ belonging to the ball centered at $\boldsymbol{p}$:

$$\boldsymbol{\nu}_l[\boldsymbol{p}, \phi] = \frac{\sum_{\boldsymbol{q} \in \mathcal{D}} K(\boldsymbol{q} - \boldsymbol{p})\delta\left(\phi_{\boldsymbol{q}}, l\right) \boldsymbol{I}_{\boldsymbol{q}}}{\sum_{\boldsymbol{q} \in \mathcal{D}} K(\boldsymbol{q} - \boldsymbol{p})\delta\left(\phi_{\boldsymbol{q}}, l\right)}$$

Label modification at pixel $\boldsymbol{v}$ yields the following variation, which can naturally be split into two components:

$$\begin{aligned} \Delta_{\boldsymbol{v}} J\left[\phi\right] = &\ \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\nu}_m[\boldsymbol{v}, \phi']\right\|^2 - \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\nu}_i[\boldsymbol{v}, \phi]\right\|^2 \\ &+ \sum_{\boldsymbol{p} \in \mathcal{D} \setminus \{\boldsymbol{v}\}} \left\|\boldsymbol{I}_{\boldsymbol{p}} - \boldsymbol{\nu}_{\phi_{\boldsymbol{p}}}[\boldsymbol{p}, \phi']\right\|^2 - \left\|\boldsymbol{I}_{\boldsymbol{p}} - \boldsymbol{\nu}_{\phi_{\boldsymbol{p}}}[\boldsymbol{p}, \phi]\right\|^2 \end{aligned}$$

where the first terms is the direct consequence from label change at $\boldsymbol{v}$. The second term, summed over the whole image domain except $\boldsymbol{v}$, results from the addition or removal of $\boldsymbol{I}_{\boldsymbol{v}}$ in affected local means. According to:

$$\boldsymbol{\nu}_l[\boldsymbol{p}, \phi'] = \begin{cases} \dfrac{\sum_{\boldsymbol{q} \in \mathcal{B}_r(\boldsymbol{p})} K(\boldsymbol{q} - \boldsymbol{p})\delta\left(\phi'_{\boldsymbol{q}}, l\right) \boldsymbol{I}_{\boldsymbol{q}}}{\sum_{\boldsymbol{q} \in \mathcal{B}_r(\boldsymbol{p})} K(\boldsymbol{q} - \boldsymbol{p})\delta\left(\phi'_{\boldsymbol{q}}, l\right)} \\ \qquad \text{if } \boldsymbol{p} \in \mathcal{B}_r(\boldsymbol{v}) \text{ and } (l = i \text{ or } l = m) \\ \boldsymbol{\nu}_l[\boldsymbol{p}, \phi] \ \text{ otherwise} \end{cases} \tag{2}$$

local means are modified only in the ball of radius $r = 3\sigma$ surrounding $\boldsymbol{v}$. Thus, the sum of variations in the second term can be restricted within this ball:

$$\begin{aligned} \Delta_{\boldsymbol{v}} J\left[\phi\right] = &\ \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\nu}_m[\boldsymbol{v}, \phi']\right\|^2 - \left\|\boldsymbol{I}_{\boldsymbol{v}} - \boldsymbol{\nu}_i[\boldsymbol{v}, \phi]\right\|^2 \\ &+ \sum_{\boldsymbol{p} \in \mathcal{B}_r(\boldsymbol{v}) \setminus \{\boldsymbol{v}\}} \left\|\boldsymbol{I}_{\boldsymbol{p}} - \boldsymbol{\nu}_{\phi_{\boldsymbol{p}}}[\boldsymbol{p}, \phi']\right\|^2 - \left\|\boldsymbol{I}_{\boldsymbol{p}} - \boldsymbol{\nu}_{\phi_{\boldsymbol{p}}}[\boldsymbol{p}, \phi]\right\|^2 \end{aligned}$$

Moreover, the modification of local means does not apply to pixels which labels are different from i or m, which further simplifies the implementation of the sum in the previous equation.

3 Non-parametric probability function

We detail here the calculus of variations for the non-parametric probability modeling term addressed in section 3.3 ”Non-parametric probability modeling”.We can express energy variation for the removal of pixel $\boldsymbol{v}$ from region $\mathcal{R}_i$ and its addition to region $\mathcal{R}_m$ with the following expression:

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & \sum_{\boldsymbol{p}\in\mathcal{D}} -\log\ \mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi']\,\delta\left(\phi'_{\boldsymbol{p}}, i\right) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} -\log\ \mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi']\,\delta\left(\phi'_{\boldsymbol{p}}, m\right) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi]\,\delta\left(\phi_{\boldsymbol{p}}, i\right) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi]\,\delta\left(\phi_{\boldsymbol{p}}, m\right)\end{aligned}$$

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & -\sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi']\,\delta\left(\phi_{\boldsymbol{p}}, i\right) + \log \mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{v}}, \phi'] \\ & - \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi']\,\delta\left(\phi_{\boldsymbol{p}}, m\right) - \log \mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{v}}, \phi'] \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi]\,\delta\left(\phi_{\boldsymbol{p}}, i\right) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi]\,\delta\left(\phi_{\boldsymbol{p}}, m\right)\end{aligned}$$

Simplifications gives the following discrete variation:

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & -\log\ \frac{\mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{v}}, \phi']}{\mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{v}}, \phi']} \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \frac{\mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi]}{\mathrm{p}_i[\boldsymbol{I}_{\boldsymbol{p}}, \phi']}\delta\left(\phi_{\boldsymbol{p}}, i\right) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log\ \frac{\mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi]}{\mathrm{p}_m[\boldsymbol{I}_{\boldsymbol{p}}, \phi']}\delta\left(\phi_{\boldsymbol{p}}, m\right)\end{aligned}$$

A direct computation of this variation is expensive. In particular, the complexity is in the computation of the two last terms. However, we reduce the complexity of calculation using histogram $\mathrm{h}_l[\boldsymbol{a}, \phi]$ and the Parzen window-based histogram $\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]$ based on $\mathrm{p}_l[\boldsymbol{a}, \phi]$.

$$\begin{aligned}\mathrm{h}_l[\boldsymbol{a}, \phi] &= \sum_{\boldsymbol{q}\in\mathcal{D}} \delta\left(\boldsymbol{a}, \boldsymbol{I}_{\boldsymbol{q}}\right)\delta\left(\phi_{\boldsymbol{q}}, l\right) \\ \hat{\mathrm{h}}_l[\boldsymbol{a}, \phi] &= \sum_{\boldsymbol{q}\in\mathcal{D}} K_\sigma\left(\boldsymbol{a} - \boldsymbol{I}_{\boldsymbol{q}}\right)\delta\left(\phi_{\boldsymbol{q}}, l\right)\end{aligned}$$

Using histograms, we can express variations in the following way:

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & -\log \frac{\mathrm{p}_m[\boldsymbol{I_v}, \phi']}{\mathrm{p}_i[\boldsymbol{I_v}, \phi']} \\ & + |\mathcal{R}_i| \log \frac{|\mathcal{R}_i|+1}{|\mathcal{R}_i|} + |\mathcal{R}_m| \log \frac{|\mathcal{R}_m|+1}{|\mathcal{R}_m|} \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log \frac{\hat{\mathrm{h}}_i[\boldsymbol{I_p}, \phi]}{\hat{\mathrm{h}}_i[\boldsymbol{I_p}, \phi']} \delta(\phi_{\boldsymbol{p}}, i) \\ & + \sum_{\boldsymbol{p}\in\mathcal{D}} \log \frac{\hat{\mathrm{h}}_m[\boldsymbol{I_p}, \phi]}{\hat{\mathrm{h}}_m[\boldsymbol{I_p}, \phi']} \delta(\phi_{\boldsymbol{p}}, m)\end{aligned}$$

With a view to simplification, the sum of log-values in the histogram can be written as:

$$\begin{aligned}\sum_{\boldsymbol{p}\in\mathcal{R}_l} \log \hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi] = & \sum_{\boldsymbol{p}\in\mathcal{R}_l} \sum_{\boldsymbol{a}\in\mathcal{A}} \log \hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi] \delta(\boldsymbol{I_p}, \boldsymbol{a}) \\ = & \sum_{\boldsymbol{a}\in\mathcal{A}} \log \hat{\mathrm{h}}_l[\boldsymbol{a}, \phi] \sum_{\boldsymbol{p}\in\mathcal{R}_l} \delta(\boldsymbol{I_p}, \boldsymbol{a}) \\ = & \sum_{\boldsymbol{a}\in\mathcal{A}} \mathrm{h}_l[\boldsymbol{a}, \phi] \log \hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]\end{aligned}$$

so

$$\begin{aligned}\sum_{\boldsymbol{p}\in\mathcal{R}_l} \log \frac{\hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi]}{\hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi']} = & \sum_{\boldsymbol{p}\in\mathcal{R}_l} \log \hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi] - \log \hat{\mathrm{h}}_l[\boldsymbol{I_p}, \phi'] \\ = & \sum_{\boldsymbol{a}\in\mathcal{A}} \mathrm{h}_l[\boldsymbol{a}, \phi] \log \frac{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi']}\end{aligned} \tag{3}$$

Since $K_{\boldsymbol{\sigma}}(x)$ can be considered as negligible as x exceeds 3σ, a simplification of the derivative in eq. (3) consists in limiting the explored range in the color neighborhood of $\boldsymbol{I_v}$:

$$N_\sigma(\boldsymbol{I_v}) = \{\boldsymbol{a} \in \mathcal{A} \mid \|\boldsymbol{I_v} - \boldsymbol{a}\| < 3\sigma\}$$

Indeed, histogram bins for colors far enough from $\boldsymbol{I_v}$ are not affected by the addition or removal of $\boldsymbol{I_v}$. Trivially, for these colors, we have

$$\log \frac{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi']} \approx 0 \text{ if } \boldsymbol{a} \notin N_\sigma(\boldsymbol{I_v}),$$

which allows to write

$$\sum_{\boldsymbol{a}\in\mathcal{A}} \mathrm{h}_l[\boldsymbol{a}, \phi] \log \frac{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi']} \approx \sum_{\boldsymbol{a}\in N_\sigma(\boldsymbol{I_v})} \mathrm{h}_l[\boldsymbol{a}, \phi] \log \frac{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_l[\boldsymbol{a}, \phi']}$$

With this simplification, the variation becomes

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & -\log \frac{\mathrm{p}_m[\boldsymbol{I_v}, \phi']}{\mathrm{p}_i[\boldsymbol{I_v}, \phi']} \\ & + |\mathcal{R}_i| \log \frac{|\mathcal{R}_i|+1}{|\mathcal{R}_i|} + |\mathcal{R}_m| \log \frac{|\mathcal{R}_m|+1}{|\mathcal{R}_m|} \\ & + \sum_{\boldsymbol{a} \in N_\sigma(\boldsymbol{I_v})} \mathrm{h}_i[\boldsymbol{a}, \phi] \log \frac{\hat{\mathrm{h}}_i[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_i[\boldsymbol{a}, \phi']} \\ & + \sum_{\boldsymbol{a} \in N_\sigma(\boldsymbol{I_v})} \mathrm{h}_m[\boldsymbol{a}, \phi] \log \frac{\hat{\mathrm{h}}_m[\boldsymbol{a}, \phi]}{\hat{\mathrm{h}}_m[\boldsymbol{a}, \phi']}\end{aligned}$$

4 Regularization term

Having formulated our framework in terms of a generic energy measure, we now introduce specific calculus of regularization energy variation. From our original definition :

$$J[\phi] = \underbrace{\sum_{\boldsymbol{p} \in \mathcal{D}} f[\phi_{\boldsymbol{p}}, \boldsymbol{p}, \phi]}_{\text{Data term}} + \underbrace{\lambda \sum_{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C}} u(\boldsymbol{p}, \boldsymbol{q}) \left(1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right)}_{\text{Regularization term}}$$

we recall the smoothness term :

$$J[\phi] = \sum_{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C}} 1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})$$

The calculus of variations is directly obtained with:

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] &= \sum_{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C}} 1 - \delta\left(\phi'_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}\right) - \sum_{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C}} 1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}) \\ &= \sum_{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C}} \left[1 - \delta\left(\phi'_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}\right)\right] - \left[1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right]\end{aligned}$$

$$\begin{aligned}\Delta_{\boldsymbol{v}} J[\phi] = & \sum_{\substack{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C} \\ \boldsymbol{p} \neq \boldsymbol{v}, \boldsymbol{q} \neq \boldsymbol{v}}} \left[1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right] - \left[1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right] \\ & + \sum_{\substack{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C} \\ \boldsymbol{p} = \boldsymbol{v}, \boldsymbol{q} \neq \boldsymbol{v}}} \left[1 - \delta\left(\phi'_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}\right)\right] - \left[1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right] \\ & + \sum_{\substack{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C} \\ \boldsymbol{p} \neq \boldsymbol{v}, \boldsymbol{q} = \boldsymbol{v}}} \left[1 - \delta\left(\phi_{\boldsymbol{p}}, \phi'_{\boldsymbol{q}}\right)\right] - \left[1 - \delta(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}})\right] \\ & + \sum_{\substack{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C} \\ \boldsymbol{p} = \boldsymbol{v}, \boldsymbol{q} = \boldsymbol{v}}} \left[1 - \delta(\phi'_{\boldsymbol{v}}, \phi_{\boldsymbol{v}})\right] - \left[1 - \delta(\phi_{\boldsymbol{v}}, \phi_{\boldsymbol{v}})\right]\end{aligned}$$

In the last equation, the first and last terms vanish. Therefore, we can express variations by:

$$
\begin{aligned}
\Delta_{\boldsymbol{v}} J\left[\phi\right] &= 2 \sum_{\substack{(\boldsymbol{p},\boldsymbol{q}) \in \mathcal{C} \\ \boldsymbol{p}=\boldsymbol{v}, \boldsymbol{q} \neq \boldsymbol{v}}} \left[1-\delta\left(\phi'_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}\right)\right] - \left[1-\delta\left(\phi_{\boldsymbol{p}}, \phi_{\boldsymbol{q}}\right)\right] \\
&= 2 \sum_{\boldsymbol{q} \in N(\boldsymbol{v}) \setminus \{\boldsymbol{v}\}} \left[1-\delta\left(\phi'_{\boldsymbol{v}}, \phi_{\boldsymbol{q}}\right)\right] - \left[1-\delta\left(\phi_{\boldsymbol{v}}, \phi_{\boldsymbol{q}}\right)\right]
\end{aligned}
$$