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Hyperquaternions: a New Tool for Physics

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Abstract A hyperquaternion formulation of Clifford algebras in n dimensions is presented. The hyperquaternion algebra is defined as a tensor product of quaternion algebras $\mathbb{H}$ (or a subalgebra thereof). An advantage of this formulation is that the hyperquaternion product is defined independently of the choice of the generators. The paper gives an explicit expression of the generators and develops a generalized multivector calculus. Due to the isomorphism $\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R})$, hyperquaternions yield all real, complex and quaternion square matrices. A hyperconjugation is introduced which generalizes the concepts of transposition, adjunction and transpose quaternion conjugate. As applications, simple expressions of the unitary and unitary symplectic groups are obtained. Finally, the hyperquaternions are compared, in the context of physical applications, to another algebraic structure based on octonions which has been proposed recently.

Keywords Clifford algebras · hyperquaternions · quaternions · hyperconjugation

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1 Introduction

Clifford algebras have become a major mathematical tool in physics and have been widely developed in recent years [1–13]. Clifford algebras allow the definition of spinors in n dimensions and give a good representation of the pseudo-orthogonal groups [14]. Well-known examples of Clifford algebras are the Pauli and Dirac algebras. Though these algebras are generally expressed in terms of complex matrices, another representation is possible.

This paper presents a hyperquaternion formulation of Clifford algebras. Defining hyperquaternions as a tensor product of quaternion algebras (or a subalgebra thereof), it follows that hyperquaternions are Clifford algebras due to a theorem by Clifford. An advantage of this formulation is that the hyperquaternion product is defined independently of the choice of the generators. Recently, we have applied such algebras to $3D$ classical and $4D$ relativistic physics obtaining, in particular, simple expressions of the rotation groups [10, 11]. Here, we propose to go farther and to apply hyperquaternions to nD physics. We give an explicit expression of the generators, develop a generalized multivector calculus and introduce a hyperconjugation yielding simple expressions of the unitary and unitary symplectic groups. Finally, we relate this new tool to physics and to another octonionic approach.

2 Background: quaternions and Clifford algebras

Quaternions [11] denoted $\mathbb{H}$ constitute a set of four real numbers

$$a = a_0 + a_1 i + a_2 j + a_3 k \tag{1}$$

where i, j, k multiply according to

$$i^2 = j^2 = k^2 = ijk = -1 \tag{2}$$
$$ij = -ji = k \tag{3}$$
$$jk = -kj = i \tag{4}$$
$$ki = -ik = j. \tag{5}$$

Two quaternions multiply according to the rule

$$\begin{aligned} ab = (&a_0b_0 - a_1b_1 - a_2b_2 - a_3b_3) \\ &+(a_0b_1 + a_1b_0 + a_2b_3 - a_3b_2)i \\ &+(a_0b_2 + a_2b_0 + a_3b_1 - a_1b_3)j \\ &+(a_0b_3 + a_3b_0 + a_1b_2 - a_2b_1)k. \end{aligned} \tag{6}$$

The conjugate of a quaternion is defined by

$$a_c = a_0 - a_1 i - a_2 j - a_3 k. \tag{7}$$

with

$$aa_c = a_0^2 + a_1^2 + a_2^2 + a_3^2 \tag{8}$$

$$(ab)_c = b_c a_c. \tag{9}$$

Clifford algebras are defined as follows and satisfy the Clifford theorem.

Definition 1 Clifford's algebra C_n is defined as an algebra (over $\mathbb{R}$) composed of n generators $e_1, e_2, ..., e_n$ multiplying according to the rule $e_i e_j = -e_j e_i$ $(i \neq j)$ and such that $e_i^2 = \pm 1$. The algebra C_n contains 2^n elements constituted by the n generators, the various products $e_i e_j, e_i e_j e_k, ...$ and the unit element 1.

Definition 2 The even subalgebra C_n^+ is generated by the products of an even number of generators: $e_i e_j, e_i e_j e_k e_m$, etc. $(i \neq j \neq k \neq m)$. The rest of the algebra contains the products of an odd number of terms and is called the odd part C_n^-.

Examples of Clifford algebras (over $\mathbb{R}$) are

- Complex numbers $\mathbb{C}$ $\left(e_1 = i, e_1^2 = -1\right)$.
- Quaternions $\mathbb{H}$ $\left(e_1 = j, e_2 = k, e_i^2 = -1\right)$.
- Biquaternions $\mathbb{C} \otimes \mathbb{H} \simeq \mathbb{H} \otimes \mathbb{C}$ $\left(e_1 = iI, e_2 = iJ, e_3 = iK, e_i^2 = 1\right)$ which are isomorphic to the Pauli algebra.
- Tetraquaternions $\mathbb{H} \otimes \mathbb{H}$ $\left(e_0 = j, e_1 = kI, e_2 = kJ, e_3 = kK, e_0^2 = -1, e_1^2 = e_2^2 = e_3^2 = 1\right)$.
- Dirac algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$

The small i, j, k represent the first quaternionic system ($i = i \otimes 1$, etc.) and the capital I, J, K the second one ($I = 1 \otimes i$, etc.), both systems commuting with each other.

Clifford's theorem relates all Clifford algebras to tensor products of quaternion algebras (or a subalgebra thereof).

Theorem 1 *If $n = 2m$ (m : integer), the Clifford algebra C_{2m} is the tensor product of m quaternion algebras. If $n = 2m - 1$, the Clifford algebra C_{2m-1} is the tensor product of $m - 1$ quaternion algebras and the algebra $(1, \omega)$ where ω is the product of the $2m$ generators ($\omega = e_1 e_2 ... e_{2m}$) of the algebra C_{2m}.*

Proof The above examples prove the theorem up to $n = 4$. For any n, one proceeds by recurrence [15, p. 378]. Suppose the theorem is true for $C_{2(n-1)}$, one adds to $C_{2(n-1)}$ the quantities

$$p = e_1 e_2 ... e_{2(n-1)} e_{2n-1}, q = e_1 e_2 ... e_{2(n-1)} e_{2n} \tag{10}$$

which anticommute among themselves and commute with the elements of $C_{2(n-1)}$; thus they form a quaternionic system commuting with $C_{2(n-1)}$. The basis of C_{2n} is obtained by the various products of the elements of $C_{2(n-1)}$ with p, q which proves the theorem. □

3 Hyperquaternion formalism

In this section, we define the hyperquaternions, the hyperquaternion product and give an explicit expression of the generators in n dimensions. Then, we present the basic concepts of a generalized multivector calculus.

3.1 Products

We shall define the hyperquaternions as a tensor product of quaternion algebras (or a subalgebra thereof). The term hyperquaternions was used by Moore to designate Lipschitz's algebras, isomorphic to Clifford algebras [16]. The hyperquaternion product is simply the product in a tensor product of quaternion algebras.

For example, a general element Q of $\mathbb{H} \otimes \mathbb{H}$ i.e. a tetraquaternion, can be viewed as a quaternion having quaternions as coefficients

$$Q = q_0 + q_1 i + q_2 j + q_3 k = (q_0; q_1, q_2, q_3), \; q_i \in \mathbb{H}. \tag{11}$$

The product of two tetraquaternions Q, P is defined by

$$QP = \begin{pmatrix} q_0p_0 - q_1p_1 - q_2p_2 - q_3p_3; q_0p_1 + q_1p_0 + q_2p_3 - q_3p_2, \\ q_0p_2 + q_2p_0 + q_3p_1 - q_1p_3, q_0p_3 + q_3p_0 + q_1p_2 - q_2p_1 \end{pmatrix}. \tag{12}$$

Similarly, a general element of $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ is a quaternion having tetraquaternions as coefficients

$$R = r_0 + r_1 i + r_2 j + r_3 k = (r_0; r_1, r_2, r_3), \; r_i \in \mathbb{H} \otimes \mathbb{H} \tag{13}$$

leading to an immediately operational product.

The product in the even subalgebras such as $\mathbb{H} \otimes \mathbb{C}$, $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$ is defined similarly. It is to be noticed that the hyperquaternion product is defined independently of any specific choice of the generators of the Clifford algebra.

3.2 Generators

The $2n$ generators e_α $(1 \leq \alpha \leq 2n)$ of $C_{2n} = \mathbb{H} \otimes \mathbb{H} ... \otimes \mathbb{H}$ (n times) can be chosen in various ways. We shall adopt the choice

$$e_1 = j \otimes 1 \otimes ... \otimes 1 \;\; (n \text{ terms}) \tag{14}$$
$$e_2 = k \otimes i \otimes 1 ... \otimes 1 \tag{15}$$
$$e_3 = k \otimes j \otimes 1 ... \otimes 1 \tag{16}$$
$$...$$
$$e_{2\beta} = k \otimes ... \otimes k \otimes i \otimes 1 ... \otimes 1 \tag{17}$$
$$e_{2\beta+1} = k \otimes ... \otimes k \otimes j \otimes 1 ... \otimes 1 \tag{18}$$
$$...$$
$$e_{2n} = k \otimes k ... \otimes k \tag{19}$$

Table 1 Generators and signature of hyperquaternions

Algebra	$C_{p,q}$	Generators
$\mathbb{C}$	$C_{0,1}$	i
$\mathbb{H}$	$C_{0,2}$	j, k
$\mathbb{C}\otimes\mathbb{H}$	$C_{3,0}$	iI, iJ, iK
$\mathbb{H}\otimes\mathbb{H}$	$C_{3,1}$	j, kI, kJ, kK
$\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}$	$C_{2,3}$	iI, iJ, IKl, iKm, iKn
$\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$	$C_{2,4}$	j, kI, kJ, kKl, kKm, kKn
$\mathbb{C}\otimes\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$	$C_{5,2}$	$(iI, iJ, iKl, iKm, iKnL,$ $iKnM, iKnN)$
$\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$	$C_{5,3}$	$(j, kI, kJ, kKl, kKm,$ $kKnL, kKnM, kKnN)$

where i, j stand at the $(\beta+1)$th place from the left with $0<\beta<n-1$. These generators anticommute among themselves and square to ± 1.

As to the generators of the algebra C_{2n-1} which is the even subalgebra of C_{2n} , they can be defined as

$$f_1 = e_1e_2, f_2 = e_1e_3, ...f_{2n-1} = e_1e_{2n}. \tag{20}$$

Adopting the notation $C_{p,q}$ $(p+q=n)$ where p is the number of generators squaring to $+1$ and q the number of those squaring to -1, the generators and signature of a few hyperquaternions are given in Table 1. The small i, j, k stand for the first quaternionic system, the capital I, J, K for the second one, l, m, n for the third one ($l = 1\otimes 1\otimes i\otimes 1$, etc.) and the capital L, M, N for the fourth one ($L = 1\otimes 1\otimes 1\otimes i$, etc.); all distinct quaternionic systems commuting with each other. Though hyperquaternions have a definite signature, another signature might be obtained by complexifying the generators. Hence, hyperquaternions yield all Clifford algebras. The explicit expression of the generators in the general case constitutes a direct second proof of Clifford's theorem.

3.3 Generalized multivector calculus

Once the generators have been chosen, the hyperquaternion algebra C_n is uniquely structured into multivectors. The 2^n elements of the algebra are constituted by scalars ($s\in V_0$), vectors ($e_i\in V_1$), bivectors ($e_ie_j\in V_2, i\neq j$), trivectors ($e_ie_je_k\in V_3, i\neq j\neq k$), etc. and a pseudo-scalar ($e_1e_2...e_n\in V_n$) where V_i are the multivector spaces. As an example, the algebra $C_4\simeq\mathbb{H}\otimes\mathbb{H}$ has the multivector structure indicated in Table 2.

The conjugate A_c of a general element A is defined as an antiinvolution obtained by replacing the e_i by their opposite $-e_i$ and reversing the order of the elements $e_ie_je_k\to e_ke_je_i$, etc.. Hence,

$$(A_c)_c = A, (AB)_c = (B_c)(A_c). \tag{21}$$

Table 2 Multivector structure of $\mathbb{H} \otimes \mathbb{H}$

1	$I = e_3 e_2$	$J = e_1 e_3$	$K = e_2 e_1$
$i = e_0 e_1 e_2 e_3$	$iI = e_0 e_1$	$iJ = e_0 e_2$	$iK = e_0 e_3$
$j = e_0$	$jI = e_0 e_3 e_2$	$jJ = e_0 e_1 e_3$	$jK = e_0 e_2 e_1$
$k = e_1 e_2 e_3$	$kI = e_1$	$kJ = e_2$	$kK = e_3$

The dual of A (denoted A^*) is

$$A^* = iA \tag{22}$$

with $i = e_1 \wedge e_2 \wedge ... \wedge e_n$ (the exterior product being defined below) and the commutator of two hyperquaternions is

$$[A, B] = \frac{1}{2}(AB - BA). \tag{23}$$

The generalized interior and exterior products of two vectors a $(= \Sigma_{i=1}^n a_i e_i)$, b, respectively denoted by $a.b$ and $a \wedge b$ can be defined, up to a sign, by the identity [15, p. 362]

$$ab = \frac{\lambda}{2\lambda}(ab + ba) + \frac{\mu}{2\mu}(ab - ba) \tag{24}$$

$$= \lambda a.b + \mu a \wedge b \tag{25}$$

where λ, μ are constant coefficients equal to ± 1. Adopting the choice $\lambda = \mu = \pm 1$ and postulating the relations $a.b = b.a$, $a \wedge b = -b \wedge a$, one finds

$$a.b = \frac{1}{2\lambda}(ab + ba) \in V_0, \;\; a \wedge b = \frac{1}{2\lambda}(ab - ba) \in V_2. \tag{26}$$

The introduction of the coefficient λ allows to eventually change the sign of the metric. A multivector $A_p = v_1 \wedge v_2 \wedge ... \wedge v_p$ $(2 \leq p \leq n)$ where v_i are vectors, is then defined by recurrence [17]

$$a.A_p = \frac{1}{2\lambda^p}\left[aA_p - (-1)^p A_p a\right] \in V_{p-1} \tag{27}$$

$$a \wedge A_p = \frac{1}{2\lambda^p}\left[aA_p + (-1)^p A_p a\right] \in V_{p+1}. \tag{28}$$

By definition, one adopts

$$A_p.a = (-1)^{p-1} \, a.A_p \tag{29}$$

$$A_p \wedge a = (-1)^p \, a \wedge A_p. \tag{30}$$

A major property of the exterior product is its associativity.

The products between multivectors A_p and B_q $(p \leq q)$ are then defined by

$$A_p \cdot B_q \equiv (v_1 \wedge v_2 \wedge \cdots \wedge v_{p-1}) \cdot (v_p \cdot B_q) \tag{31}$$

$$A_p \wedge B_q \equiv v_1 \wedge (v_2 \wedge \cdots \wedge v_p \wedge B_q) \tag{32}$$

with

$$A_p \cdot B_q = (-1)^{p(q+1)} B_q \cdot A_p. \tag{33}$$

4 Hyperquaternions, matrices and hyperconjugation

In this section we relate hyperquaternions to real, complex and quaternion matrices. The concept of hyperconjugation is introduced which generalizes the concepts of transposition, adjunction and transpose quaternion conjugate.

To establish the connection of hyperquaternions with matrices, consider the linearly independent matrices

$$m_{i\otimes 1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, m_{j\otimes 1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, m_{k\otimes 1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$m_{1\otimes i} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, m_{1\otimes j} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, m_{1\otimes k} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}.$$

These matrices square to -1, anticommute on the same line and commute with the matrices of the other line, hence, they constitute distinct quaternionic systems. The products of these matrices generate $m(4,\mathbb{R})$, and thus we get the isomorphisms

$$\mathbb{H} \otimes \mathbb{H} \simeq m(4,\mathbb{R}) \tag{34}$$

$$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{C} \simeq m(4,\mathbb{C}) \tag{35}$$

$$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{H} \simeq m(4,\mathbb{H}) \tag{36}$$

indicating that hyperquaternions yield all real matrices as well as the complex and quaternionic ones (see Appendix A).

Furthermore, one can notice that a transposition of the above antisymmetric matrices corresponds to a quaternionic conjugation. This allows to define a hyperconjugation as

$$A^H = \mathbb{H}_c \otimes \mathbb{H}_c \otimes \mathbb{H}_c \otimes ... \otimes \mathbb{H}_c \tag{37}$$

where A^H denotes the hyperconjugate of a general element A of the algebra $\mathbb{H}\otimes \mathbb{H}\otimes\mathbb{H}\otimes...\otimes\mathbb{H}$ and c indicates a quaternion conjugation. The hyperconjugation yields respectively the matrix transposition

$$\mathbb{H}_c \otimes \mathbb{H}_c \simeq [m(4,\mathbb{R})]^t\,, \tag{38}$$

the adjunction

$$\mathbb{H}_c \otimes \mathbb{H}_c \otimes \mathbb{C}_c \simeq [m(4,\mathbb{C})]^\dagger\,, \tag{39}$$

and the transpose quaternion conjugate

$$\mathbb{H}_c \otimes \mathbb{H}_c \otimes \mathbb{H}_c \simeq [m(4,\mathbb{H})]^t_c\,. \tag{40}$$

These properties will allow us to obtain simple hyperquaternion expressions of the unitary and unitary symplectic groups as we shall see below.

5 Unitary and unitary symplectic groups

As application of the hyperconjugation concept, we consider the unitary and unitary symplectic groups. The importance of the unitary groups is well known. The unitary symplectic group is related to the symplectic groups which are relevant in Quantum Mechanics [6].

5.1 Unitary and special unitary groups SU(n)

To define the unitary group $U(n)$, consider the algebra A

$$A = \mathbb{H} \otimes \mathbb{H} \otimes ...\mathbb{H} \{2p \text{ terms}\} \otimes \mathbb{C} \simeq m(4^p, \mathbb{C}) \tag{41}$$

where p is an integer. Since, as we have seen above, the hyperconjugation corresponds to the adjunction (transpose, complex conjugate), the unitary group $U(n)$ with $n = 4^p$, can be defined as the elements U of A such that

$$U^H = U^{-1}. \tag{42}$$

The special unitary group $SU(n)$ corresponds to elements U such that $\det U = 1$ and has $n^2 - 1$ generators g_i with

$$U = e^{g_i \theta}. \tag{43}$$

As a specific example, take $SU(4)$ with $p = 1$ and the algebra $[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{C} \simeq m(4, \mathbb{C})$. The 15 generators g_i of $SU(4)$ can be read off directly from the basis of $\mathbb{H} \otimes \mathbb{H}$ of Table 2, with

$$g_i \in \left\{ \begin{array}{c} i, j, k, I, J, K, i'iI, i'iJ, i'iK, \\ i'jI, i'jJ, i'jK, i'kI, i'kJ, i'kK \end{array} \right\} \tag{44}$$

where i' $(= l)$ is the ordinary complex imaginary and

$$e^{I\theta} = \cos\theta + I \sin\theta \tag{45}$$

$$e^{i'iI\theta} = \cos\theta + i'iI \sin\theta \tag{46}$$

$$\left(e^{I\theta}\right)^H = e^{-I\theta} = \left(e^{I\theta}\right)^{-1} \text{etc.}. \tag{47}$$

If the elements g_i are represented by matrices $\widehat{g_i}$, one notices that the latter are traceless hence,

$$\det \widehat{U} = \det e^{\widehat{g_i}\theta} = e^{(\text{trace}\widehat{g_i})\theta} = 1. \tag{48}$$

Thus the elements $U = e^{g_i \theta}$ constitute a concise hyperquaternionic expression of the group $SU(4)$. As subgroups, one has $SU(2)$ and $SU(3)$ (see Appendix B).

5.2 Unitary symplectic groups USp(n)

To define the unitary symplectic group $USp(n)$, one proceeds similarly with the algebra A

$$A = \mathbb{H} \otimes \mathbb{H} \otimes ...\mathbb{H}\,\{2p \text{ terms}\} \otimes \mathbb{H} \simeq m(4^p, \mathbb{H}) \tag{49}$$

where p is an integer. The hyperconjugation corresponding to the transpose, quaternion conjugate, the unitary symplectic group $USp(n)$ with $n = 4^p$, can be defined as the elements U of A such that

$$U^H = U^{-1}. \tag{50}$$

The group $USp(n)$ has $n(2n+1)$ generators g_i with

$$U = e^{g_i \theta} \tag{51}$$

and satisfies the property

$$USp(n) = U(2n) \cap Sp(2n, \mathbb{C}) \tag{52}$$

where $Sp(2n, \mathbb{C})$ is the symplectic group [18, p. 444].

As a concrete example, consider $USp(4)$ with the algebra $A = [\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{H} \simeq m(4, \mathbb{H})$. The 36 generators g_i can be read off directly from the basis of $\mathbb{H} \otimes \mathbb{H}$ of Table 2

$$g_i \in \left\{ \begin{matrix} i', j', k', i, j, k, I, J, K, (i', j', k')iI, ()\,iJ, ()\,iK, \\ ()\,jI, ()\,jJ, ()\,jK, ()\,kI, ()\,kJ, ()\,kK \end{matrix} \right\} \tag{53}$$

where the parentheses stand for (i', j', k') with $i'(=l), j'(=m), k'(=n)$ and

$$e^{j' iI\theta} = \cos\theta + j' iI \sin\theta \tag{54}$$

$$\left(e^{j' iI\theta}\right)^H = e^{-j' iI\theta} = \left(e^{j' iI\theta}\right)^{-1}, etc.. \tag{55}$$

As subgroups, one has $USp(1)$, $USp(2)$ and $USp(3)$ (see Appendix C).

6 Relations of hyperquaternions and physics

The relations of hyperquaternions and physics are summarized in Table 3 which shows the relevance of Clifford algebras in physics. Complex numbers, quaternions and biquaternions give an excellent description of respectively $1D$, $2D$ and $3D$ classical physics [10]. The algebra $\mathbb{H} \otimes \mathbb{H}$ yields the special theory of relativity, classical electromagnetism and the general theory of relativity [11]. Dirac's algebra, isomorphic to the algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$, entails the relativistic quantum mechanics. Since the above algebras already yield much of physics, further theoretical progress seems likely to be obtained only by raising the dimension of the Clifford algebra. Hence, one might reasonably expect new theoretical results to come out of the algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$ and the associated

Table 3 Hyperquaternion physics (STR: special theory of relativity, GTR: general theory of relativity, QM: quantum mechanics)

Algebra	$C_{p,q}$	Symmetry group	Physics
$\mathbb{C}$			$1D$ physics
$\mathbb{H}$	$C_{0,2}$	$SO(2)$	$2D$ physics
$\mathbb{H} \otimes \mathbb{C} \simeq m(2, \mathbb{C})$	$C_{3,0}$	$SO(3)$	$3D$ physics
$\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R})$	$C_{3,1}$	$SO(1,3)$	STR, GTR
$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{C} \simeq m(4, \mathbb{C})$	$C_{2,3}$	$SU(4)$	relativistic QM
$[\mathbb{H} \otimes \mathbb{H}] \otimes \mathbb{H} \simeq m(4, \mathbb{H})$	$C_{2,4}$	$USp(4)$	quaternionic QM
		$\simeq U(8) \cap Sp(8, \mathbb{C})$	embedding of GTR
$[\mathbb{H} \otimes \mathbb{H}] \otimes [\mathbb{H} \otimes \mathbb{C}]$	$C_{5,2}$	$SU(8)$	
$\simeq m(8, \mathbb{C})$		$\Rightarrow SU(5)$	standard model

group $USp(4)$. If one goes over to the algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$, one obtains the group $SU(8)$ and $SU(5)$ as a subgroup which accounts for the standard model.

Other algebraic structures have been proposed recently. One of them uses a tensor product of division algebras $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ where $\mathbb{O}$ represents the octonion algebra [8]. Though this algebra is neither a Clifford algebra nor associative, it seems to give good results concerning the standard model. Furthermore, it shares with the hyperquaternion approach the idea that physical reality might be an image of an algebraic structure. Do we live in an octonionic or hyperquaternionic world? The answer can only come out of the convergence of theory with experiment.

7 Conclusion

The paper has presented a hyperquaternion calculus which yields all Clifford algebras as well as the real, complex and quaternion matrices. A hyperconjugation has been introduced which generalizes the concepts of transposition, adjunction and transpose quaternion conjugate. As applications, simple expressions of the unitary and unitary symplectic groups have been obtained. Finally, the paper has related hyperquaternions to physics and shown that hyperquaternions constitute a new, efficient, unifying tool for many applications of physics.

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A Isomorphism $\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R})$

The well-known isomorphism of $\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R})$ can be demonstrated in various ways leading to different representations [7]. To justify the representation here adopted, we provide a complete proof. Start from the Pauli matrices (with i' the usual complex imaginary)

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \sigma_2 = \begin{bmatrix} 0 & -i' \\ i' & 0 \end{bmatrix}, \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, 1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since the Pauli algebra is isomorphic to the biquaternions, one can introduce the matrices h_i

$$\sigma_1 = i'h_1, \sigma_2 = i'h_2, \sigma_3 = i'h_3$$

$$h_1 = \begin{bmatrix} 0 & -i' \\ -i' & 0 \end{bmatrix}, h_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, h_3 = \begin{bmatrix} -i' & 0 \\ 0 & i' \end{bmatrix}.$$

The matrices h_i satisfy the relations

$$h_1^2 = h_2^2 = h_3^2 = h_1 h_2 h_3 = -1$$

and thus constitute a quaternionic system isomorphic to $\mathbb{H}$. One then considers the real matrices

$$\begin{aligned} m_{1\otimes 1} &= 1 \otimes 1, m_{i\otimes 1} = i'h_1 \otimes h_2, m_{j\otimes 1} = h_2 \otimes 1, m_{k\otimes 1} = i'h_3 \otimes ih_2 \\ m_{1\otimes 1} &= 1 \otimes 1, m_{1\otimes i} = h_2 \otimes i'h_1, m_{1\otimes j} = 1 \otimes h_2, m_{1\otimes k} = h_2 \otimes i'h_3 \end{aligned}$$

(in [7, p. 55], one takes $m_{j\otimes 1} = -h_2 \otimes 1$, $m_{1\otimes j} = -1 \otimes h_2$).

The products of these matrices generate 16 real matrices constituting a basis of $m(4, \mathbb{R})$ which is thus isomorphic to $\mathbb{H} \otimes \mathbb{H}$. Explicitly, one has

$$m_{i\otimes 1} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, m_{j\otimes 1} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, m_{k\otimes 1} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$m_{1\otimes i} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, m_{1\otimes j} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, m_{1\otimes k} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}$$

$$m_{i\otimes i} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, m_{i\otimes j} = \begin{bmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, m_{i\otimes k} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$$

$$m_{j\otimes i} = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}, m_{j\otimes j} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, m_{j\otimes k} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$m_{k\otimes i} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, m_{k\otimes j} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, m_{k\otimes k} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

together with the unit matrix

$$m_{1\otimes 1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

As a consequence

$$[\mathbb{H} \otimes \mathbb{H}] \otimes [\mathbb{H} \otimes \mathbb{H}] \simeq m(4, \mathbb{R}) \otimes m(4, \mathbb{R}) \simeq m(16, \mathbb{R}), etc..$$

Hence, any real square matrix can be expressed in terms of hyperquaternions.

B Special unitary groups: SU(2), SU(3) and SU(4)

The isomorphism $[\mathbb{H}\otimes\mathbb{H}]\otimes\mathbb{C}\simeq m(4,\mathbb{C})$ entails that any complex matrix $m(4,\mathbb{C})$ can be expressed in terms of complex hyperquaternions $[\mathbb{H}\otimes\mathbb{H}]\otimes\mathbb{C}$. One just needs to solve 32 linear equations. Applying this procedure, we express the matrices $\widehat{g_i}=i'\widehat{\lambda_i}$, where $\widehat{\lambda_i}$ are the standard Gell-Mann matrices [19, pp. 388-389], in terms of complex hyperquaternions g_i. We thus obtain the hyperquaternion generators $e^{g_i\theta}$ of the unitary subgroups.

B.1 $SU(2)$

$$g_1=\frac{i'}{2}\left(iK-jI\right),g_2=\frac{-1}{2}\left(J+k\right),g_3=\frac{-i'}{2}\left(iI+jK\right)$$

$$\widehat{g_1}=\begin{bmatrix}0&i'&0&0\\i'&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\widehat{g_2}=\begin{bmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix},\widehat{g_3}=\begin{bmatrix}i'&0&0&0\\0&-i'&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}.$$

B.2 $SU(3)$

To the generators of $SU(2)$ one adds

$$g_4=\frac{i'}{2}\left(-iJ+kI\right),g_5=\frac{-1}{2}\left(K+j\right),g_6=\frac{-i'}{2}\left(jJ+kK\right)$$

$$g_7=\frac{-1}{2}\left(I-i\right),g_8=\frac{i'}{2\sqrt{3}}\left(-iI+jK-2kJ\right)$$

$$\widehat{g_4}=\begin{bmatrix}0&0&i'&0\\0&0&0&0\\i'&0&0&0\\0&0&0&0\end{bmatrix},\widehat{g_5}=\begin{bmatrix}0&0&1&0\\0&0&0&0\\-1&0&0&0\\0&0&0&0\end{bmatrix},\widehat{g_6}=\begin{bmatrix}0&0&0&0\\0&0&i'&0\\0&i'&0&0\\0&0&0&0\end{bmatrix}$$

$$\widehat{g_7}=\begin{bmatrix}0&0&0&0\\0&0&1&0\\0&-1&0&0\\0&0&0&0\end{bmatrix},\widehat{g_8}=\frac{1}{\sqrt{3}}\begin{bmatrix}i'&0&0&0\\0&i'&0&0\\0&0&-2i&0\\0&0&0&0\end{bmatrix}.$$

B.3 $SU(4)$

To the generators of $SU(3)$ one adds

$$g_9=\frac{i'}{2}\left(jJ-kK\right),g_{10}=\frac{-1}{2}\left(i+I\right),g_{11}=\frac{-i'}{2}\left(iJ+kI\right)$$

$$g_{12}=\frac{-1}{2}\left(j-K\right),g_{13}=\frac{-i'}{2}\left(iK+jI\right),g_{14}=\frac{1}{2}\left(k-J\right)$$

$$g_{15}=\frac{i'}{\sqrt{6}}\left(iI-jK-kJ\right)$$

$$\widehat{g_9}=\begin{bmatrix}0&0&0&i'\\0&0&0&0\\0&0&0&0\\i'&0&0&0\end{bmatrix},\widehat{g_{10}}=\begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\-1&0&0&0\end{bmatrix},\widehat{g_{11}}=\begin{bmatrix}0&0&0&0\\0&0&0&i'\\0&0&0&0\\0&i'&0&0\end{bmatrix}$$

$$\widehat{g_{12}}=\begin{bmatrix}0&0&0&0\\0&0&0&1\\0&0&0&0\\0&-1&0&0\end{bmatrix},\widehat{g_{13}}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&i'\\0&0&i'&0\end{bmatrix},\widehat{g_{14}}=\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&1\\0&0&-1&0\end{bmatrix}$$

$$\widehat{g_{15}}=\frac{1}{\sqrt{6}}\begin{bmatrix}i'&0&0&0\\0&i'&0&0\\0&0&i'&0\\0&0&0&-3i'\end{bmatrix}.$$

C Unitary symplectic groups: USp(1), USp(2) and USp(3)

We use the same procedure as above using the isomorphism $[\mathbb{H}\otimes\mathbb{H}]\otimes\mathbb{H}\simeq m(4,\mathbb{H})$. The $n\,(2n+1)$ generators of $USp(n)$ are $e^{g_i\theta}$ with the following hyperquaternionic g_i.

C.1 $USp(1)$

The 3 elements are

$$g_i\in\left\{i'(=l),j'(=m),k'(=n)\right\}.$$

C.2 $USp(2)$

The 10 elements are besides those of $USp(1)$ and with $()=(i',j',k')$

$$g_i\in\left\{\frac{1}{2}(i',j',k')\,(iK-jI)\,,\frac{-1}{2}\,(J+k)\,,\frac{-1}{2}\,()\,(iI+jK)\right\}.$$

C.3 $USp(3)$

The 21 elements are besides those of $USp(2)$

$$g_i\in\left\{\begin{array}{c}\frac{1}{2}(i',j',k')\,(-iJ+kI)\,,\frac{-1}{2}\,(j+K)\,,\frac{-1}{2}\,()\,(jJ+kK)\,,\\ \frac{-1}{2}\,(I-i)\,,\frac{-1}{2\sqrt{3}}\,()\,(-iI+jK-2kJ)\end{array}\right\}.$$

C.4 $USp(4)$

The 36 elements are besides those of $USp(3)$

$$g_i\in\left\{\begin{array}{c}\frac{1}{2}(i',j',k')\,(jJ-kK)\,,\frac{-1}{2}\,(i+I)\,,\frac{-1}{2}\,()\,(iJ+kI)\,,\\ \frac{-1}{2}\,(j-K)\,,\frac{-1}{2}\,()\,(iK+jI)\,,\frac{1}{2}\,(k-J)\,,\\ \frac{1}{\sqrt{6}}\,()\,(iI-jK-kJ)\end{array}\right\}.$$

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