Dual Hyperquaternion Poincaré Groups

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Abstract. A new representation of the Poincaré groups in n dimensions via dual hyperquaternions is developed, hyperquaternions being defined as a tensor product of quaternion algebras (or a subalgebra thereof). This formalism yields a uniquely defined product and simple expressions of the Poincaré generators, with immediate physical meaning, revealing the algebraic structure independently of matrices or operators. An extended multivector calculus is introduced (allowing an eventual sign change of the metric or of the exterior product). The Poincaré groups are formulated as a dual extension of hyperquaternion pseudo-orthogonal groups. The canonical decomposition and the invariants are discussed. As concrete example, the $4D$ Poincaré group is examined together with a numerical application. Finally, the hyperquaternion representation is compared to the quantum mechanical and the octonionic ones. Potential applications include in particular, moving reference frames and computer graphics.

Mathematics Subject Classification (2010). Primary 15A66; Secondary 11E88,15A67.

Keywords. Quaternions, Hyperquaternions, Dual hyperquaternions, Poincaré groups, Canonical decomposition.

1. Introduction

The $4D$-Poincaré group is the group of linear transformations leaving invariant the Lorentz metric $ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$ (with the signature $+ - --$) and is constituted of rotations and space-time translations. This group is of great relevance in physics in particular, general relativity, relativistic quantum mechanics and particle physics, free particles being characterized by invariants of that group [1]. An important subgroup of the Poincaré group is the group of euclidean motions including rotations and translation in $3D$ euclidean space. Two major methods for physical applications have been developed in that case, the homogeneous matrix transform and dual quaternions [18, 28]. Generalizing the Poincaré group to a pseudo-euclidean space in n dimensions (with an arbitrary signature), we shall call them Poincaré groups. Various representations of the Poincaré groups have been proposed either in specific dimensions or signatures and are often expressed in terms of matrices [4, 5, 7, 8, 16, 19, 21, 24, 25, 26, 27]. Yet, matrices are not the only nor probably the best way to represent rotation groups. An alternative is to use Clifford algebras in particular hyperquaternions defined as a tensor product of quaternion algebras (or a subalgebra thereof). Recently, we have applied hyperquaternions to the unitary, unitary symplectic and pseudo-orthogonal groups in n dimensions and have briefly expressed the Poincaré groups via dual hyperquaternions [9, 10]. Here, we propose to develop in more details this representation which is a dual hyperquaternion Clifford algebra extension of $3D$ dual quaternions. The method gives simple expressions of the Poincaré generators, reveals their algebraic nature and provides a compact, efficient computation distinct from the matrix one. After a short introduction specifying the basic concepts and notation, an extended multivector calculus is presented (allowing an eventual sign change of the metric or of the exterior product). Then we discuss the nD Poincaré group, its algebra, a canonical decomposition into simple planes and the invariants. As concrete example, we study the $4D$ Poincaré group, provide a numerical example and compare the hyperquaternion representation to the quantum mechanical one. Potential applications include in particular, moving reference frames and computer graphics.

2. Background: Hyperquaternion Algebras

We briefly introduce hyperquaternions to specify the notations and basic concepts [9, 10, 12]. The quaternion algebra $\mathbb{H}$ is constituted by quaternions

$$a = a_1 + a_2 i + a_3 j + a_4 k \qquad (a_i \in \mathbb{R}) \tag{2.1}$$

where i, j, k satisfy the fundamental relations

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, etc.. \tag{2.2}$$

The quaternion product is given by

$$\begin{aligned} ab &= (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) && (2.3)\\ &\quad +i\,(a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3) && (2.4)\\ &\quad +j\,(a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4) && (2.5)\\ &\quad +k\,(a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)\,. && (2.6) \end{aligned}$$

The quaternion conjugate is $a_c = a_1 - a_2 i - a_3 j - a_4 k$ with

$$aa_c = a_1^2 + a_2^2 + a_3^2 + a_4^2, (ab)_c = b_c a_c. \tag{2.7}$$

A hyperquaternion is a tensor product of quaternion algebras (or a subalgebra thereof). Thus, $\mathbb{H} \otimes \mathbb{H}$ is the tensor product of two quaternion algebras.

Calling (i, j, k) and (I, J, K) two commuting quaternionic systems, one writes

$$(i, j, k) \otimes 1 = (i, j, k), \quad 1 \otimes (i, j, k) = (I, J, K) \tag{2.8}$$

which uniquely specifies the tensor product. To define $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$, one introduces a third quaternionic system (l, m, n) commuting with the previous ones. Similarly, one obtains $\mathbb{H} \otimes \mathbb{H} \otimes ... \otimes \mathbb{H}$ (and the subalgebras $\mathbb{H} \otimes \mathbb{C}$, $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$, etc.). Due to the isomorphism $\mathbb{H} \otimes \mathbb{H} \simeq m(4, \mathbb{R})$, hyperquaternions yield all square real, complex and quaternionic matrices. A hyperconjugation $\mathbb{H}_c \otimes \mathbb{H}_c \otimes ... \otimes \mathbb{H}_c$ entails the transposition, adjunction and the transposition quaternion conjugate [10].

Whereas Hamilton viewed quaternions as a $3D$ (if not $4D$) system, Clifford, adopting Grassmann's ideas, considered quaternions as having only 2 generators ($e_1 = i, e_2 = j, e_1 e_2 = k, e_1^2 = e_2^2 = -1$) suitable for a $2D$ plane physics. He furthermore was the first to introduce tensor products of quaternion algebras, the concept of tensor product ("compounds of algebras") having been introduced a few years earlier. In his fundamental paper, Clifford proved that tensor products of quaternions constitute Clifford algebras [3]. A proof and an explicit expression of the generators is given in [10]. Lipschitz, shortly after and independently of Clifford, gave a simple expression of the n-dimensional euclidean rotation groups and thereby rediscovered the (even) Clifford algebra [20]. Moore, was to call Lipschitz's algebras hyperquaternions and developed a canonical decomposition (into simple orthogonal planes) thereof [22, 23]. An extension of Moore's method to pseudo-euclidean rotations has recently been presented by the authors [9]. An advantage of the hyperquaternion formalism over the matrix one, is to yield physically meaningful parameters and straightforward computations. Moreover, besides rotations, hyperquaternions yield all unitary and unitary symplectic groups [10]. Mathematically, hyperquaternions (defined as tensor products of quaternion algebras) are Clifford algebras $C_n(p, q)$ having $n = p + q$ generators e_i such that $e_i e_j + e_j e_i = 0$ $(i \neq j)$, $e_i^2 = +1$ (p generators) and $e_i^2 = -1$ (q generators) where the generators are given in a compact hyperquaternionic form (for example $e_1 = iKl$, etc.). Products of distinct generators yield multivectors V_n such as vectors e_i (V_1), bivectors $e_i e_j$ (V_2), trivectors $e_i e_j e_k$ (V_3) etc.. C^+ is the (even) subalgebra constituted by products of an even number of e_i, C^- is the rest of the algebra. The conjugate A_c of a general element A is obtained by replacing the e_i by their opposite $-e_i$ and reversing the order of the elements

$$(A_c)_c = A, (AB)_c = (B_c)(A_c). \tag{2.9}$$

The commutator of two hyperquaternions is $[A, B] = \frac{1}{2}(AB - BA)$ and the dual of A is $A^* = i_d A$ where $i_d = e_1 \wedge e_2 ... \wedge e_n$ (to be defined in the next section). The operations between the multivectors constitute the multivector calculus which we shall now examine.

3. Extended Multivector Calculus

As compared to the standard multivector calculus [2], we present here an extended multivector calculus allowing an eventual change of sign of the metric or the exterior product [10, 11, 12, 13].

The interior and exterior products of two vectors a $(= \sum_{1}^{n} a_i e_i)$, b can be defined via the identity

$$
\begin{aligned}
2ab &= \lambda\lambda^{-1}(ab+ba) + \mu\mu^{-1}(ab-ba) && (3.1)\\
&= \lambda a.b + \mu a \wedge b && (3.2)
\end{aligned}
$$

where λ, μ are two constant coefficients equal to ± 1 (making possible a change of sign of the metric or of the exterior product); thus,

$$2a.b = \lambda^{-1}(ab+ba), 2a \wedge b = \mu^{-1}(ab-ba). \tag{3.3}$$

Next, we consider products between a vector and a multivector. Given a multivector $A_p = a_1 \wedge a_2 \wedge ... \wedge a_p$ $(2 \leq p < n)$ where a_p are vectors, we define the interior product [2]

$$a.A_p = \sum_{k=1}^{p} (-1)^{k+1} (a.a_k)\, a_1 \wedge ... \wedge a_{k-1} \wedge a_{k+1} \wedge a_{k-1} ... \wedge a_p. \tag{3.4}$$

The particular multivectors $a \wedge A_2, a \wedge A_3$ are defined via the relations

$$aA_2 = \frac{\lambda}{\mu}(a.A_2) + a \wedge A_2, aA_3 = \lambda(a.A_3) + \mu a \wedge A_3. \tag{3.5}$$

Generalized to a multivector A_p $(2 \leq p < n)$, the above relations become

$$
\begin{aligned}
aA_p &= \frac{\lambda}{\mu^{p-1}}(a.A_p) + \mu^p a \wedge A_p && (3.6)\\
A_p a &= \frac{\lambda}{\mu^{p-1}}(A_p.a) + \mu^p A_p \wedge a. && (3.7)
\end{aligned}
$$

Postulating, a priori

$$A_p.a \equiv (-1)^{p-1} a.A_p, A_p \wedge a \equiv (-1)^p a \wedge A_p, \tag{3.8}$$

one derives from Eq.[3.7] after multiplication by $(-1)^p$

$$(-1)^p A_p a = \frac{-\lambda}{\mu^{p-1}}(a.A_p) + \mu^p a \wedge A_p. \tag{3.9}$$

Combining Eqs.[3.6-3.9], the general formulas yield

$$
\begin{aligned}
2a.A_p &= \mu^{p-1}\lambda^{-1}\left[aA_p - (-1)^p A_p a\right] && (3.10)\\
2a \wedge A_p &= \mu^{-p}\left[aA_p + (-1)^p A_p a\right] && (3.11)
\end{aligned}
$$

(giving the standard formulas for $\lambda = \mu = 1$).

Interior and exterior products between multivectors are defined by

$$
\begin{aligned}
A_p \wedge B_q &= a_1 \wedge (a_2 \wedge ... \wedge a_p \wedge B_q) && (3.12)\\
A_p.B_q &= (a_1 \wedge ... \wedge a_{p-1}).(a_p.B_q), \quad (p \leq q) && (3.13)
\end{aligned}
$$

with $A_p.B_q = (-1)^{p(q+1)} B_q.A_p$ [2]. In particular, for bivectors B_i one has

$$B_1 B_2 = B_1.B_2 + B_1 \wedge B_2 + [B_1, B_2] \tag{3.14}$$

yielding respectively a scalar, a tetravector and a bivector. These relations constitute the basic computational rules of the hyperquaternion algebras which we shall now apply to the Poincaré groups.

4. Poincaré Groups in n Dimensions

In this section, we develop a hyperquaternion representation of the Poincaré group in n dimensions. To this effect, we embed the nD space in an affine $(n+1)\,D$ space and express the rotations, reflections and translations of the Poincaré group as rotations and reflections in the affine space. We begin with the algebraic formalism followed by the canonical decomposition and the invariants.

4.1. Algebraic Formalism

Consider a hyperquaternion algebra $C_{n+1}(p', q')$ having $n+1$ $(= p' + q')$ generators (squaring to ± 1) $e_1, e_2, ...e_n, e_{n+1}$ and let X be an element of an affine space

$$X = e_{n+1} + \varepsilon x \tag{4.1}$$

where x belongs to the vector space V_1 with $x = \sum_{i=1}^{n} e_i x_i$ $(x_i \in \mathbb{R})$ and ε commutes with all generators $\left(\varepsilon^2 = 0\right)$. The hyperquaternion algebra $C_n(p, q)$ associated with V_1 $(n = p + q)$ has the metric (with $\lambda = \mu = 1$)

$$\begin{aligned} ds^2 &= dx.dx = dx^2 && (4.2)\\ &= \left(dx_1^2 + ... + dx_p^2\right) - \left(dx_{p+1}^2 + ... + dx_{p+q}^2\right). && (4.3) \end{aligned}$$

A vector x is timelike if $x.x > 0$, spacelike if $x.x < 0$ and isotropic if $x.x = 0$. The Poincaré group of V_1 are the isometries of this metric constituted by the pseudo-orthogonal group $O\,(p, q)$ and translations which we shall consider successively.

The pseudo-orthogonal group $O\,(p, q)$ is generated by at most n orthogonal symmetries. An orthogonal symmetry with respect to a plane (going through the origin) and perpendicular to a unit vector u $\left(u^2 = \pm 1\right)$ is expressed by the formula (see Appendix A)

$$x' = \frac{uxu}{uu_c} \tag{4.4}$$

with $x'^2 = x^2$, $uu_c = -u^2$. Hence, time and space like symmetries correspond respectively to

$$x' = -uxu \quad \left(u^2 = 1\right), \; x' = uxu \quad \left(u^2 = -1\right). \tag{4.5}$$

Combining r time and s space symmetries one obtains the four types of pseudo-orthogonal transformations A of $O\,(p, q)$ as indicated in Table 1. Sub-

Table 1. Hyperquaternion group $O(p,q)$ with r time and s space symmetries (e: even, o: odd)

component	$L_+^\uparrow$	$L_+^\downarrow$	$L_-^\uparrow$	$L_-^\downarrow$
(r,s)	(e,e)	(o,o)	(e,o)	(o,e)
$det\ A$	1	1	-1	-1
$x' =$	axa_c	$-axa_c$	$-axa_c$	axa_c
$aa_c =$	1	-1	1	-1
$a \in$	$C_n^+(p,q)$	$C_n^+(p,q)$	$C_n^-(p,q)$	$C_n^-(p,q)$

groups of $O(p,q)$ are [27]

$$O(p,q) = L_+^\uparrow \oplus L_+^\downarrow \oplus L_-^\uparrow \oplus L_-^\downarrow \tag{4.6}$$

$$SO^+(p,q) = L_+^\uparrow,\ SO(p,q) = L_+^\uparrow \oplus L_+^\downarrow. \tag{4.7}$$

Furthermore,

$$L_-^\downarrow = e_1 L_+^\uparrow, L_-^\uparrow = e_{p+1} L_+^\uparrow, L_+^\downarrow = e_1 e_{p+1} L_+^\uparrow \tag{4.8}$$

where e_1, e_{p+1} can be replaced by other unit vectors of the same type. Thus, one has

$$a = e_1 a' \ \left(aa_c = -1, a'a'_c = 1, a \in C_n^-(p,q), a' \in C_n^+(p,q)\right)\ etc. \tag{4.9}$$

Embedding $C_n(p,q)$ in the algebra $C_{n+1}(p',q')$, the $O(p,q)$ group leaves the axis e_{n+1} unchanged and can be expressed as

$$X' = aXa_c = e_{n+1} + \varepsilon x' \tag{4.10}$$

with $x' = \pm axa_c \ \left(aa_c = \pm 1, a \in C_{n+1}^+(p',q') \text{ or } C_{n+1}^-(p',q')\right)$.

A translation T (in V_n) is given by

$$X' = bXb_c \ = e_{n+1} + \varepsilon\,(x+t) \tag{4.11}$$

with

$$b = e^{\varepsilon e_{n+1}\frac{t}{2}} = 1 + \varepsilon e_{n+1}\frac{t}{2} \ \left(bb_c = 1, b \in C_{n+1}^+(p',q'), e_{n+1}^2 = -1\right) \tag{4.12}$$

and $t = \sum_{i=1}^n e_i t_i$, $t_i \in \mathbb{R}$ (if $e_{n+1}^2 = 1$, one takes $b = e^{\varepsilon \frac{t}{2} e_{n+1}}$). Since $bb_c = 1$ and $b \in C_{n+1}^+(p',q')$, a translation corresponds to a rotation of SO_{n+1}^+.

Combining the pseudo-orthogonal group $O(p,q)$ and the translations T, one obtains the full Poincaré group of Table 2 with the relations

$$P = P_+^\uparrow \oplus P_+^\downarrow \oplus P_-^\uparrow \oplus P_-^\downarrow \tag{4.13}$$

$$P_-^\uparrow = e_1 P_+^\uparrow, P_-^\downarrow = e_{p+1} P_+^\uparrow, P_+^\downarrow = e_1 e_{p+1} P_+^\uparrow \tag{4.14}$$

where $P_+^\uparrow$ is the restricted Poincaré group; for its Lie algebra, see Appendix B. Our next step will be the canonical decomposition of the restricted Poincaré group.

TABLE 2. Hyperquaternion Poincaré group

component	$P_+^\uparrow$	$P_+^\downarrow$	$P_-^\uparrow$	$P_-^\downarrow$
$det\ A$	1	1	-1	-1
$X' =$	fXf_c	$-fXf_c$	$-fXf_c$	fXf_c
$ff_c =$	1	-1	1	-1
$f \in$	$C_{n+1}^+(p',q')$	$C_{n+1}^+(p',q')$	$C_{n+1}^-(p',q')$	$C_{n+1}^-(p',q')$

4.2. Canonical Decomposition of the Restricted Group

An element of the restricted Poincaré group $P_+^\uparrow$ being a rotation of SO_{n+1}^+, one can apply the canonical decomposition of pseudo-orthogonal rotations presented in [9]. To this effect, consider the algebra $C_{n+1}(p',q')$ having $n+1$ generators with $n = 2k$ (even) or $2k+1$ (odd) and the Poincaré transform

$$X' = fXf_c \ (ff_c = 1, f \in C_{n+1}^+ (p',q')). \tag{4.15}$$

The even (dual) hyperquaternion f is of the type

$$f = S + P + \frac{P \wedge P}{2S} + ... \tag{4.16}$$

where P is a (dual) bivector. From f one computes

$$B = \frac{P}{S} = M + \varepsilon N \tag{4.17}$$

with

$$N = e_{n+1} \sum_{i=1}^{n} e_i \alpha_i \left(\alpha_i \in \mathbb{R}\right), \tag{4.18}$$

where N is a simple plane ($N \wedge N = 0$) since all terms contain the vector e_{n+1}. The canonical decomposition $B = \sum_{i=1}^m b_i B_i$ yields at most $m = k+1$ orthogonal simple (dual) planes B_i

$$B_i = M_i + \varepsilon N_i, \quad B_i^2 \in \{\pm 1, 0\}. \tag{4.19}$$

From $B_i \wedge B_i = 0$, one obtains

$$\begin{aligned} & (M_i + \varepsilon N_i) \wedge (M_i + \varepsilon N_i) & (4.20)\\ = \ & M_i \wedge M_i + 2\varepsilon N_i \wedge M_i = 0 & (4.21)\end{aligned}$$

where we have used the commutativity of the exterior product of two bivectors; hence,

$$M_i \wedge M_i = 0 \ , \ N_i \wedge M_i = 0 \tag{4.22}$$

which entails that M_i is a simple plane and that N_i belongs to the same plane and anticommutes with it ($N_i M_i = -M_i N_i$). For $M_i \neq 0$, one has

$$\begin{aligned} B_i^2 &= (M_i + \varepsilon N_i)(M_i + \varepsilon N_i) & (4.23)\\ &= M_i^2 = \pm 1 & (4.24)\end{aligned}$$

which for $B_i^2 = -1$ ($b_i = \tan\frac{\Phi_i}{2}$) yields

$$e^{\frac{\Phi_i}{2}B_i} = \cos\frac{\Phi_i}{2} + (M_i + \varepsilon N_i)\sin\frac{\Phi_i}{2}; \tag{4.25}$$

and for $B_i^2 = 1$ ($b_i = \tanh\frac{\Phi_i}{2}$)

$$e^{\frac{\Phi_i}{2}B_i} = \cosh\frac{\Phi_i}{2} + (M_i + \varepsilon N_i)\sinh\frac{\Phi_i}{2}. \tag{4.26}$$

For $M_i = 0$, one has a pure translation $e^{\varepsilon N_i} = 1 + \varepsilon N_i$. Finally, one obtains the algebraically compact decomposition

$$f = e^{\frac{\Phi_1}{2}B_1} e^{\frac{\Phi_2}{2}B_2} ... e^{\frac{\Phi_m}{2}B_m}. \tag{4.27}$$

For each component $f_i = e^{\frac{\Phi_i}{2}B_i}$, the rotation $R_i = e^{\frac{\Phi_i}{2}M_i}$ is known. Writing

$$f_i = R_i T_i \quad (\text{or } T_i R_i\ ) \tag{4.28}$$

the translation T_i is obtained as $T_i = R_i^{-1} f_i$ (or $f_i R_i^{-1}$). For the entire f, one has $f = f_1 f_2 ... f_m$ where the f_i commute, hence,

$$\begin{aligned} f &= (R_1T_1)(R_2T_2)...(R_mT_m) && (4.29)\\ &= (R_1R_2...R_m)(T_1T_2...T_m) = RT && (4.30)\end{aligned}$$

yielding the translation $T = R^{-1}f$. In the same way, one obtains

$$\begin{aligned} f &= (T_1R_1)(T_2R_2)...(T_mR_m) && (4.31)\\ &= (T_1T_2...T_m)(R_1R_2...R_m) = TR && (4.32)\end{aligned}$$

and finally $T = fR^{-1}$.

4.3. Invariants of the Restricted Group

The Poincaré invariants of the restricted group $P_+^\uparrow$ are obtained as follows. The intersection of the simple plane N with the space V_n $(e_1e_2...e_n)$ is a vector P, parallel to N giving the invariant P^2. Next, we consider the multivectors

$$\begin{aligned} W_1 &= P \wedge (M + \varepsilon N) = P \wedge M && (4.33)\\ W_2 &= P \wedge b_1(M_1 + \varepsilon N_1) \wedge b_2(M_2 + \varepsilon N_2) && (4.34)\\ &= (b_1b_2)\, P \wedge M_1 \wedge M_2 && (4.35)\\ &\ ... && (4.36)\\ W_{k-1} &= (b_1b_2...b_{k-1})\, P \wedge M_1 \wedge M_2 \wedge ... \wedge M_{k-1} && (4.37)\end{aligned}$$

yielding the invariant inner products

$$(W_1.W_1), (W_2.W_2), ..., (W_{k-1}.W_{k-1}). \tag{4.38}$$

If the dimension of the space is even $(n = 2m)$, one thus obtains $k-1$ invariants and with P^2, a total of k invariants. If the dimension of the space is odd $(n = 2k+1)$, one has the k invariants above plus the pseudo-scalar

$$W_m = (b_1b_2...b_k)\, P \wedge M_1 \wedge M_2 \wedge ... \wedge M_k \tag{4.39}$$

which is an invariant by itself. As concrete example, we shall now examine the $4D$ Poincaré group.

5. Example: 4D Poincaré group

The $4D$-Poincaré group is of central importance in physics, in particular in relativistic quantum mechanics and general relativity [1, 14, 15]. We shall first present the algebraic formulation, then a numerical application with a canonical decomposition and the invariants. Finally, we shall compare the hyperquaternion representation with the quantum mechanical operator representation.

5.1. Algebraic Formulation

Consider the hyperquaternion algebra $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{C}$ ($\simeq C_{2,3}$) having five generators (see Appendix C)

$$e_0 = iJ, e_1 = iKl, e_2 = iKm, e_3 = iKn, e_4 = iI. \tag{5.1}$$

The metric of the algebra $C_{1,3}$ ($e_4 = 0$) is

$$ds^2 = dx.dx = dx^2 = dx_0^2 - dx_1^2 - dx_2^2 - dx_3^2 \tag{5.2}$$

$\left(x = \sum_{i=0}^{3} e_i x_i,\ x_i \in \mathbb{R}\right)$. The restricted Poincaré group $P_+^\uparrow$ is composed of spatial rotations, hyperbolic rotations (boosts) and space-time translations which are respectively given by a total of ten generators (each of the two first equations below yield three generators and the third one four)

$$\begin{aligned}
e^{\frac{\theta}{2}B} &= \cos\frac{\theta}{2} + \sin\frac{\theta}{2}B \quad \left[B^2 = -1, B \in (l,m,n)\right] && (5.3)\\
e^{\frac{\Phi}{2}B} &= \cosh\frac{\Phi}{2} + \sinh\frac{\Phi}{2}B \quad \left[B^2 = 1, B \in (Il,Im,In)\right] && (5.4)\\
e^{\varepsilon\frac{\lambda}{2}B} &= 1 + \varepsilon\frac{\lambda}{2}B \quad \left[B^2 = (\pm 1, 0), B \in (K,Jl,Jm,Jn), \lambda \in \mathbb{R}\right]. && (5.5)
\end{aligned}$$

The combination of these transformations generate the element f

$$X' = fXf_c \quad \left(ff_c = 1, f \in C_{2,3}^{+}\right) \tag{5.6}$$

($X = iI + \varepsilon x$, $X' = iI + \varepsilon x'$). The canonical decomposition of f leads to at most two simple orthogonal planes B_i

$$\begin{aligned}
B &= b_1B_1 + b_2B_2 = M + \varepsilon N && (5.7)\\
f &= e^{\frac{\Phi_1}{2}B_1}e^{\frac{\Phi_2}{2}B_2} \quad \left(B_i^2 = \pm 1, 0\right). && (5.8)
\end{aligned}$$

The projection of the bivector N on the space $V_4\,(e_0e_1e_2e_3)$ gives a vector P and the invariant P^2 which can be positive, negative or nil. The second invariant is $(W_1.W_1)$ with $W_1 = P\wedge M$. In the following, we shall provide a numerical example to illustrate the procedure.

5.2. Numerical Example

As numerical example of a canonical decomposition, we consider the product of a spatial rotation followed by a translation and a boost leading to the element f of the $4D-$Poincaré transform $X' = fXf_c$

$$f = e^{\frac{\Phi_2}{2} mI} e^{\varepsilon(-2Jl+K)} e^{\frac{\Phi_1}{2} m} \tag{5.9}$$

$$= \left(2 + \sqrt{3}mI\right) \left[1 + \varepsilon\left(-2Jl + K\right)\right] \left(\frac{\sqrt{3}}{2} + \frac{m}{2}\right) \tag{5.10}$$

$\left(tan\frac{\Phi_1}{2} = \frac{1}{\sqrt{3}} = b_1, tanh\frac{\Phi_2}{2} = \frac{\sqrt{3}}{2} = b_2\right)$. The canonical decomposition leads to the expression [9]

$$f = e^{\frac{\Phi_2}{2} B_2} e^{\frac{\Phi_1}{2} B_1} \tag{5.11}$$

where Φ_1, Φ_2 have the same values as above and B_1, B_2 are two simple orthogonal commuting (dual) planes

$$B_1 = m - 2\varepsilon J\left(\sqrt{3}l + n\right), \quad B_2 = mI + \varepsilon\left(-Jm + \frac{2}{\sqrt{3}}K\right) \tag{5.12}$$

such that $B_1^2 = -1, B_2^2 = 1$. From the relation $B = M + \varepsilon N \ (= b_1 B_1 + b_2 B_2)$, one finds

$$M = m\left(\frac{1}{\sqrt{3}} + \frac{\sqrt{3}}{2}I\right), \quad N = -J\left(2l + \frac{\sqrt{3}}{2}m + \frac{2}{\sqrt{3}}n\right) + K. \tag{5.13}$$

The orthogonal projection of the plane N on the four-space $V_4 = e_1e_2e_3e_4 (= I)$ gives a vector P via the formula

$$P = N^*.V_4 \tag{5.14}$$

$$= (iN).V_4 = \frac{iNI - IiN}{2} = -iIN \tag{5.15}$$

$$= iK\left(2l + \frac{\sqrt{3}}{2}m + \frac{2}{\sqrt{3}}m\right) + iJ \tag{5.16}$$

and the invariant $P^2 = -\frac{61}{12}$. Computing

$$W_1 = P \wedge M = \frac{PM + MP}{2} \in V_3 \tag{5.17}$$

$$= i\left(-Jl + \frac{Jm}{\sqrt{3}} + \sqrt{3}iJn - \frac{K}{2}\right), \tag{5.18}$$

one obtains the invariant $W_1.W_1 = -\frac{49}{12} = P^2\left(M_\perp\right)^2$ where $M_\perp$ is the component of M perpendicular to P [12]

$$M_\perp = P^{-1}\left(P \wedge M\right) \tag{5.19}$$

$$= \frac{1}{61}\left(24l - \sqrt{3}m - 8\sqrt{3}n - 10Il + 32\sqrt{3}mI - 14\sqrt{3}In\right) \tag{5.20}$$

and $M_\perp^2 = \frac{49}{61}$. The Clifford dual of the three-vector W_1 is the vector W

$$W = IW_1 = i\left(-Kl + \frac{Km}{\sqrt{3}} + \sqrt{3}iKn + \frac{J}{2}\right) \tag{5.21}$$

yielding the same invariant $W^2 = W_1^2$ (with $I = e_0e_1e_2e_3$). The vector W plays a similar role as the Pauli-Lubanski vector in quantum mechanics. This numerical example illustrates the fact that the dual hyperquaternion formulation completely reveals the abstract algebraic properties of the Poincaré group making it perhaps more accessible than other representations.

5.3. Other Representations

Other representations of the $4D$-Poincaré group exist. In quantum mechanics, the translations and rotations (spatial and hyperbolic) are represented respectively by the operators

$$\widehat{P_\mu} = \frac{\partial}{\partial x^\mu}, \ \widehat{M_{\mu\nu}} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} \tag{5.22}$$

acting on a spin-0 wave function with the mass as invariant. For a spin-1/2 Dirac wave function, the Poincaré generators are

$$\widehat{P_\mu} = \frac{\partial}{\partial x^\mu}, \ \widehat{M_{\mu\nu}} = x^\mu \frac{\partial}{\partial x^\nu} - x^\nu \frac{\partial}{\partial x^\mu} - \frac{1}{4}\left[\gamma_\mu, \gamma_\nu\right] \tag{5.23}$$

where γ_μ are the Dirac matrices with the anticommutator $\{\gamma_\mu, \gamma_\nu\} = -2g_{\mu\nu}$ having as invariants the mass and the spin [1]. Both representations have the same Lie algebra as the hyperquaternion representation, the latter being however spin independent (see Appendix B). The hyperquaternion representation thus constitutes a new form of representation (with hyperquaternion generators) distinct from the quantum mechanical one, revealing the abstract algebraic structure of the Poincaré group. It is to be noticed that dual hyperquaternions lead for certain Poincaré groups to a unitary representations ($\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C} \simeq m(4, \mathbb{C})$) and to unitary symplectic ones for others ($\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H} \simeq m(4, H)$).

Another Poincaré representation, developed for the standard model, makes use of a tensor product of the four division algebras $\mathbb{R} \otimes \mathbb{C} \otimes \mathbb{H} \otimes \mathbb{O}$ where $\mathbb{O}$ stands for the octonion algebra which is related to quaternions [6, 17]. Though this algebra is neither a Clifford algebra nor associative, it shares with the hyperquaternionic approach the idea that physics might result from algebra and in particular from tensor products of algebras. Yet, since groups and group representations are associative, operators have to be constructed which seem to be isomorphic to the complex Clifford algebra $Cl_6(\mathbb{C})$ leading to the isomorphisms

$$\begin{aligned} Cl_6(\mathbb{C}) &\simeq m(8, \mathbb{C}) \\ &\simeq m(4, \mathbb{R}) \otimes m(2, \mathbb{C}) \\ &\simeq (\mathbb{H} \otimes \mathbb{H}) \otimes (\mathbb{H} \otimes \mathbb{C}). \end{aligned}$$

Hence, in the end, it seems that the octonionic approach is compatible with the hyperquaternionic one.

Though Poincaré groups are very important, they do not constitute the largest covariant group of physics. Thus Maxwell's equations in vacuum are covariant with respect to the conformal group which contains the Poincaré group as subgroup. Is it possible to express the conformal groups as hyperquaternions? This will be the object of a next study.

6. Conclusion

The paper has given a new dual hyperquaternion representation of the Poincaré groups in n dimensions distinct from the matrix one. The formalism yields simple expressions of the Poincaré generators, with immediate physical meaning. After the introduction of an extended multivector calculus, the algebraic formalism of the Poincaré groups has been developed as well as the canonical decomposition and invariants. As example, the $4D$-Poincaré group and a numerical example have been examined. Finally, the hyperquaternion representation has been compared to the quantum mechanical and octonionic ones. It is hoped that the dual hyperquaternion representation might deepen the abstract algebraic understanding of the Poincaré groups and provide a new compact, efficient computational tool. Potential applications include in particular, moving reference frames and computer graphics.

Acknowledgment

This work was supported by the LABEX PRIMES (ANR-11-LABX-0063) and was performed within the framework of the LABEX CELYA (ANR-10-LABX-0060) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR).

The authors gratefully acknowledge the comments of the journal referees which have greatly improved the readability of the paper.

Appendix A. Orthogonal Plane Symmetry

For the convenience of the reader, we derive here the formula of Eq. 4.4 [12, 13]. The orthogonal symmetric x' of a vector x with respect to a plane orthogonal to a vector u satisfies the equations

$$x' = x + ku \ , u.\left(\frac{x'+x}{2}\right) = 0 \ \ (k \in \mathbb{R})$$

Hence,

$$\begin{aligned} u.\left(x + \frac{ku}{2}\right) &= 0 \Rightarrow k = \frac{-2u.x}{u.u} \\ x' &= x - \frac{2\,(u.x)}{u.u} \\ &= x - \frac{(ux+xu)\,u}{uu} = \frac{uxu}{uu_c}. \end{aligned}$$

Appendix B. Lie Algebra of the nD-Poincaré Group

We first give the Lie algebra of the restricted Poincaré group $P_+^\uparrow$ and then of the full group P.

B.1. Restricted Group

Consider an nD space imbedded in an $n+1$ hyperquaternion algebra having the generators $e_1, e_2, ...e_n, e_{n+1}$. The Lie generators of the rotations are

$$M_{ij} = \frac{1}{2} e_i e_j \quad \{i, j \in [1...n], i \neq j\}.$$

The Lie commutator being defined as $[A, B] = AB - BA$, one obtains for $i \neq j = r \neq s$ and

$$\begin{aligned} [M_{ij}, M_{rs}] &= \frac{1}{4}(e_i e_j \ e_r e_s - e_r e_s \ e_i e_j) \\ &= \frac{1}{4}(e_i e_j \ e_r e_s + e_i e_r \ e_j e_s) \\ &= \frac{1}{2}\eta_{jr} \ e_i e_s = \eta_{jr} \ M_{is} \end{aligned}$$

with $\eta_{jr} = (e_j e_r + e_r e_j)/2$. Similarly, one has

$$[M_{ij}, M_{rs}] = \eta_{is} \ M_{jr} \quad (j \neq i = s \neq r)$$

$$[M_{ij}, M_{rs}] = -\eta_{js} \ M_{ir} \quad (i \neq j = s \neq r)$$

$$[M_{ij}, M_{rs}] = -\eta_{ir} \ M_{js} \quad (j \neq i = r \neq s);$$

combining all possible cases for the rotations one gets

$$[M_{ij}, M_{rs}] = \eta_{jr} \ M_{is} + \eta_{is} \ M_{jr} - \eta_{js} \ M_{ir} - \eta_{ir} \ M_{js}.$$

For the nD-translations, the generators are

$$M_{(n+1)i} = \frac{1}{2}\varepsilon e_{n+1} e_i \quad \left\{i \in [1...n], \varepsilon^2 = 0, e_{n+1}^2 = -1\right\}$$

(for $e_{n+1}^2 = 1$, the one takes $M_{i(n+1)} = -M_{(n+1)i}$). One has the relations

$$\left[M_{(n+1)i}, M_{(n+1)j}\right] = 0 \ \ (\forall i, j)$$

and for $i \neq j = k$

$$\begin{aligned} \left[M_{ij}, M_{(n+1)k}\right] &= \frac{\varepsilon}{4}\left(e_i e_j \ e_{(n+1)} e_k - e_{(n+1)} e_k \ e_i e_j\right) \\ &= \frac{\varepsilon e_{n+1}}{4}(e_i e_j \ e_k + e_k \ e_j e_i) \\ &= \frac{\varepsilon e_{n+1}}{2}\eta_{jk} \ e_i = \eta_{jk} \ M_{(n+1)i}; \end{aligned}$$

similarly, for $k = i \neq j$, one has

$$\left[M_{ij}, M_{(n+1)k}\right] = -\eta_{ik} \ M_{(n+1)j} \ .$$

Combining the two cases above, one obtains for the translations

$$\left[M_{ij}, M_{(n+1)k}\right] = \eta_{jk} M_{(n+1)i} - \eta_{ik} \ M_{(n+1)j.}$$

Projecting the plane $M_{(n+1)i}$ on the space V_n one obtains, for $e_{n+1}^2 = -1$, the vector

$$P_i = e_{n+1} M_{(n+1)i} = \frac{\varepsilon}{2} e_{n+1} e_{n+1} e_i = -\frac{\varepsilon}{2} e_i.$$

For $e_{n+1}^2 = +1$, one has

$$P_i = e_{n+1} M_{i(n+1)} = \frac{\varepsilon}{2} e_{n+1} e_i e_{n+1} = -\frac{\varepsilon}{2} e_i.$$

The complete Lie algebra of the restricted Poincaré group can thus be expressed in the standard abstract form

$$\begin{aligned} [M_{ij}, M_{rs}] &= \eta_{jr}\ M_{is} + \eta_{is}\ M_{jr} - \eta_{js}\ M_{ir} - \eta_{ir}\ M_{js} \\ [P_i, P_j] &= 0 \\ [M_{ij}, P_k] &= \eta_{jk} P_i - \eta_{ik}\ P_j. \end{aligned}$$

B.2. Full Group

The other components of the full Poincaré group being obtained from the restricted one through multiplication by a vector e_k, one has besides the above relations the following ones

$$\begin{aligned} [M_{ij}, e_k] &= \eta_{jk} e_i - \eta_{ik}\ e_j \\ \left[M_{(n+1)i}, e_k\right] &= \eta_{ik} \left(\varepsilon e_{n+1}\right). \end{aligned}$$

Appendix C. Multivector Structure of $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$

$$\begin{bmatrix} 1 & l = e_2 e_3 & m = e_3 e_1 & n = e_1 e_2 \\ I = e_0 e_1 e_2 e_3 & I\ l = e_1 e_0 & I\ m = e_2 e_0 & I\ n = e_3 e_0 \\ J = e_1 e_4 e_2 e_3 & J\ l = e_4 e_1 & J\ m = e_4 e_2 & J\ n = e_4 e_3 \\ K = e_0 e_4 & Kl = e_0 e_4 e_2 e_3 & Km = e_4 e_0 e_1 e_3 & Kn = e_0 e_4 e_1 e_2 \end{bmatrix}$$
$$+i \begin{bmatrix} 1 = e_0 e_4 e_1 e_2 e_3 & l = e_4 e_0 e_1 & m = e_4 e_0 e_2 & n = e_4 e_0 e_3 \\ I = e_4 & I\ l = e_4 e_2 e_3 & I\ m = e_1 e_4 e_3 & I\ n = e_4 e_1 e_2 \\ J = e_0 & J\ l = e_0 e_2 e_3 & J\ m = e_1 e_0 e_3 & J\ n = e_0 e_1 e_2 \\ K = e_1 e_2 e_3 & Kl = e_1 & Km = e_2 & Kn = e_3 \end{bmatrix}$$

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