Hyperquaternions: An Efficient Mathematical Formalism for Geometry

Patrick R. Girard[1 [0000-0001-8556-1816]], Patrick Clarysse[1 [0000-0002-5495-7655]], Romaric Pujol[2], Robert Goutte[1], and Philippe Delachartre[1 [0000-0003-0492-1846]]

[1] Univ Lyon, INSA-LYON, Université Claude Bernard Lyon 1, UJM-Saint Etienne, CNRS, Inserm, CREATIS UMR 5220, U1206, F-69621 LYON, France
patrick.girard@creatis.insa-lyon.fr
[2] Pôle de Mathématiques, INSA-Lyon
Bât. Léonard de Vinci, 21 avenue Jean Capelle
F-69621 Villeurbanne, France

Abstract. Hyperquaternions being defined as a tensor product of quaternion algebras (or a subalgebra thereof), they constitute Clifford algebras endowed with an associative exterior product providing an efficient mathematical formalism for differential geometry. The paper presents a hyperquaternion formulation of pseudo-euclidean rotations and the Poincaré groups in n dimensions (via dual hyperquaternions). A canonical decomposition of these groups is developed as an extension of an euclidean formalism and illustrated by a $5D$ example. Potential applications include in particular, moving reference frames and machine learning.

Keywords: Quaternions · Hyperquaternions · Pseudo-euclidean rotations · Poincaré groups · Canonical decomposition.

1 Introduction

Clifford algebras allow an excellent representation of pseudo-euclidean rotations which are important symmetry groups of physics [1–4]. A decomposition of these groups into orthogonal, commuting planar rotations is called a canonical decomposition. Various canonical decompositions have been developed which deal with either specific rotations or dimensions and are often expressed in terms of matrices [5, 6]. In a recent paper, we have introduced a hyperquaternion formulation of Clifford algebras and applied them to the unitary and unitary symplectic groups [7]. Here, we consider pseudo-euclidean rotations and the Poincaré groups in n dimensions (via dual hyperquaternions). A canonical decomposition of these groups is developed within that framework as an extension of an euclidean formalism introduced by Moore [8, 9]. After a short presentation of hyperquaternions and multivectors, we derive the pseudo-euclidean rotations and the canonical decomposition. Then we go on to the Poincaré groups and a $5D$ example. Potential applications are moving reference frames and machine learning [10]

Table 1. Biquaternion Multivector Structure

1	$i = e_3e_2$	$j = e_1e_3$	$k = e_2e_1$
$I = e_1e_2e_3$	$Ii = e_1$	$Ij = e_2$	$Ik = e_3$

2 Background: Quaternions, Hyperquaternions and Multivectors

In this section, we briefly introduce quaternions, hyperquaternions and multivectors [7, 11–15]. The quaternion algebra $\mathbb{H}$ which contains $\mathbb{R}$ and $\mathbb{C}$ as particular cases is constituted by quaternions

$$a = a_1 + a_2 i + a_3 j + a_4 k \qquad (a_i \in \mathbb{R}) \tag{1}$$

where $i,\ j, k$ multiply according to

$$i^2 = j^2 = k^2 = ijk = -1, ij = -ji = k, etc.. \tag{2}$$

The product of two quaternions a, b is given by

$$ab = (a_1b_1 - a_2b_2 - a_3b_3 - a_4b_4) + (a_1b_2 + a_2b_1 + a_3b_4 - a_4b_3)\, i \tag{3}$$
$$+ (a_1b_3 + a_3b_1 + a_4b_2 - a_2b_4)\, j + (a_1b_4 + a_4b_1 + a_2b_3 - a_3b_2)\, k. \tag{4}$$

The conjugate of a quaternion is $a_c = a_1 - a_2 i - a_3 j - a_4 k$ with

$$aa_c = a_1^2 + a_2^2 + a_3^2 + a_4^2, (ab)_c = b_c a_c \tag{5}$$

The hyperquaternion algebra (over $\mathbb{R}$) is defined as the tensor product of quaternion algebras (or a subalgebra thereof). Examples of hyperquaternion algebras are the quaternions $\mathbb{H}$, tetraquaternions $\mathbb{H} \otimes \mathbb{H}$ and so on $\mathbb{H} \otimes \mathbb{H} \otimes ... \otimes \mathbb{H}$; subalgebras are the complex numbers $\mathbb{C}$, biquaternions $\mathbb{H} \otimes \mathbb{C}$, Dirac algebra $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{C}$, etc..

Calling (i, j, k) the first quaternionic system, (I, J, K) the second one and (l, m, n) the third one, all systems commuting with each other, one has

$$i \otimes i \otimes i = iIl, \quad i \otimes j \otimes k = iJn, \ etc. \tag{6}$$

which uniquely defines the multiplication.

Hyperquaternions having n generators e_i such that $e_ie_j + e_je_i = 0$ $(i \neq j)$, $e_i^2 = \pm 1$ constitute Clifford algebras C_n. The choice of the generators entails a multivector structure as shown, in the case of biquaternions, in Table 1. The 2^n elements of the algebra are composed of scalars, vectors e_i, bivectors e_ie_j, trivectors $e_ie_je_k$ etc. yielding respectively the multivector spaces $V_0, V_1, V_2, V_3, ...V_n$. C^+ is the subalgebra constituted by products of an even number of e_i, C^- is the rest of the algebra. The multivector structure allows to define basic operations like conjugation, duality and the interior and exterior products.

Considering a general element A of the algebra, the conjugate A_c is obtained by replacing the e_i by their opposite $-e_i$ and reversing the order of the elements

$$(A_c)_c = A, (AB)_c = (B_c)(A_c). \tag{7}$$

The dual of A is $A^* = i_d A$ where $i_d = e_1 \wedge e_2 ... \wedge e_n$ (to be defined below) and the commutator of two hyperquaternions is

$$[A, B] = \frac{1}{2}(AB - BA). \tag{8}$$

The interior and exterior products of two vectors a, b are obtained as follows. From the identity

$$2ab = \lambda\lambda^{-1}[(ab + ba) + (ab - ba)] \tag{9}$$

where $\lambda = \pm 1$ is a given coefficient (allowing to eventually change the sign of the metric), one defines

$$2a.b = \lambda^{-1}(ab + ba), 2a \wedge b = \lambda^{-1}(ab - ba) \tag{10}$$

which are respectively a scalar and a bivector. A multivector $A_p = a_1 \wedge a_2 \wedge ... \wedge a_p$ $(2 \leq p < n)$ where a_p are vectors, is then defined by recurrence

$$2a.A_p = \lambda^{-p}[aA_p - (-1)^p A_p a] \in V_{p-1} \tag{11}$$

$$2a \wedge A_p = \lambda^{-p}[aA_2 + (-1)^p A_2 a] \in V_{p+1} \tag{12}$$

By definition, we take

$$A_p.a \equiv (-1)^{p-1} a.A_p, A_p \wedge a \equiv (-1)^p a \wedge A_p. \tag{13}$$

An important property of the exterior product is its associativity.

Interior and exterior products between multivectors are defined by

$$A_p \wedge B_q = a_1 \wedge (a_2 \wedge ... \wedge a_p \wedge B_q) \tag{14}$$

$$A_p.B_q = (a_1 \wedge ... \wedge a_{p-1}).(a_p.B_q), \quad (p \leq q) \tag{15}$$

with $A_p.B_q = (-1)^{p(q+1)} B_q.A_p$ [16]. In particular, we have the following useful formulas where Bi are bivectors and $V_p[A]$ the multivector part V_p of A

$$B_1 B_2 = B_1.B_2 + B_1 \wedge B_2 + [B_1, B_2] \tag{16}$$

$$B_1 \wedge B_2 = V_4[B_1 B_2] \tag{17}$$

$$B_1 \wedge B_2 \wedge B_3 = V_6[B_1(B_2 \wedge B_3)] \tag{18}$$

$$B_1.(B_2 \wedge B_3) = V_2[B_1(B_2 \wedge B_3)] \tag{19}$$

$$(B_1 \wedge B_2).(B_3 \wedge B_4 \wedge B_5) = V_2[(B_1 \wedge B_2)(B_3 \wedge B_4 \wedge B_5)]. \tag{20}$$

Hyperquaternions yield all real, complex and quaternionic square matrices as well as the transposition, adjunction and transpose quaternion conjugate via a hyperconjugation defined as $\mathbb{H}_c \otimes \mathbb{H}_c \otimes ... \otimes \mathbb{H}_c$ as indicated in Table 2.

Table 2. Hyperquaternions and matrices

$$\begin{array}{l|l} \mathbb{H}\otimes\mathbb{H}\simeq m(4,\mathbb{R}) & \mathbb{H}_c\otimes\mathbb{H}_c\simeq\ [m(4,\mathbb{R})]^t \\ \hline \mathbb{H}\otimes\mathbb{H}\otimes\mathbb{C}\simeq m(4,\mathbb{C}) & \mathbb{H}_c\otimes\mathbb{H}_c\otimes\mathbb{C}_c\simeq\ [m(4,\mathbb{C})]^\dagger \\ \mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}\simeq m(4,\mathbb{H}) & \mathbb{H}_c\otimes\mathbb{H}_c\otimes\mathbb{H}_c\simeq\ [m(4,\mathbb{H})]_c^t\,. \end{array} \tag{21}$$

3 Pseudo-Orthogonal Rotations

Here, we derive a hyperquaternion formulation of pseudo-euclidean rotations and develop a canonical decomposition. Historically, the formula of n dimensional euclidean rotations $x' = axa^{-1}$ $(a \in C_n^+)$ was given by Lipschitz [17] and Moore developed a canonical decomposition thereof [8, 9]. We introduce, as an extension of Moore's method, within the hyperquaternion Clifford algebra framework, a canonical decomposition of pseudo-euclidean rotations and the Poincaré groups. After a brief review of the basic definitions and the Cartan theorem, we develop the canonical decomposition.

3.1 Definitions and Theorem

Let $C_{p,q}$ be a hyperquaternion algebra having $n = p + q$ generators e_i and the quadratic form

$$x.y = x_1y_1 + ... + x_py_p - (x_{p+1}y_{p+1}... - x_{p+q}y_{p+q}) \tag{22}$$

$$= \lambda^{-1}(xy + yx)/2 \tag{23}$$

where x, y are vectors $(x = x_ie_i)$. A vector x is timelike if $x.x > 0$, spacelike if $x.x < 0$ and isotropic if $x.x = 0$.

An orthogonal symmetry with respect to a plane going through the origin and perpendicular to a unit vector a $\left(a^2 = \pm 1\right)$ is given by [12, 13]

$$x' = \pm axa \tag{24}$$

with $x'x' = (\pm axa)(\pm axa) = xx$.

Definition 1. *The pseudo-orthogonal group $O(p, q)$ is the group of linear operators which leave invariant the form $x \cdot y$.*

Theorem 1. *Every rotation of $O(p, q)$ is the product of an even number $2m \leq n$ of symmetries.*

Definition 2. *The special orthogonal group $SO^+(p, q)$ is constituted by rotations which preserve the orientation of the space of positive norm vectors and the space of negative norm vectors.*

A rotation of $SO^{+}(p,q)$ can thus be expressed as

$$x' = axa_c \quad (aa_c = 1) \tag{25}$$

with $a = a_1 a_2 ... a_{2m}, \in C^{+}$, where a_i are unit vectors (with an even number of timelike and spacelike vectors). Developing the product (with $\lambda = 1$)

$$a_i a_j = a_i . a_j + a_i \wedge a_j \tag{26}$$

one sees that it contains a simple plane $B = a_i \wedge a_j$ such that $B^2 = B.B + B \wedge B$ is a scalar since $B \wedge B = 0$. Hence, a rotation involves at most $m \leq n/2$ simple planes. A canonical decomposition of rotations is obtained by choosing these simple planes to be orthogonal.

3.2 Canonical Decomposition

A rotation of $SO^{+}(p,q)$ can be decomposed as

$$a = e^{\frac{\Phi_1}{2} B_1} e^{\frac{\Phi_2}{2} B_2} ... e^{\frac{\Phi_m}{2} B_m} \qquad (aa_c = 1) \tag{27}$$

where B_i are m simple orthogonal commuting planes such that $B_i^2 = \pm 1$ together for $i \neq j$

$$B_i . B_j = 0, B_i B_j = B_j B_i, B_i B_j = B_i \wedge B_j; \tag{28}$$

Φ_i are the angles of rotation within the planes B_i. According to whether $B_i^2 = -1$ or $B_i^2 = 1$, one has respectively

$$e^{\frac{\Phi_i}{2} B_i} = \cos \frac{\Phi_i}{2} + \sin \frac{\Phi_i}{2} B_i, e^{\frac{\Phi_i}{2} B_i} = \cosh \frac{\Phi_i}{2} + \sinh \frac{\Phi_i}{2} B_i. \tag{29}$$

The rotation can be developed as

$$a = S\left(1 + b_1 B_1\right)\left(1 + b_2 B_2\right) ... \left(1 + b_m B_m\right) \tag{30}$$

with $b_i = \tan \frac{\Phi_i}{2}$ (or $\tanh \frac{\Phi_i}{2}$). Since $aa_c = 1$ one has

$$S^2 \left(1 + b_1^2\right) \left(1 - b_2^2\right) \left(1 - b_3^2\right) = 1 \tag{31}$$

$$S = \frac{1}{\sqrt{\left(1 \pm b_1^2\right) ... \left(1 \pm b_m^2\right)}} \tag{32}$$

which shows that S is determined by the b_i. Writing

$$B = b_1 B_1 + b_2 B_2 + b_3 B_3 \tag{33}$$

one can express a as

$$a = S\left(1 + B + \frac{B \wedge B}{2! S^2} + ... \frac{B \wedge B \wedge B \wedge ... (m \text{ terms})}{m! S^m}\right) \tag{34}$$

which shows that the bivector B determines completely the rotation.

If the scalar is nil, for example if ($\Phi_1 = \pm\pi$, $B_1^2 = -1$), then a is proportional to B_1

$$a = B_1 e^{\frac{\Phi_2}{2} B_2} e^{\frac{\Phi_3}{2} B_3}; \tag{35}$$

one then computes $B_1^{-1}a$ and comes back to the general expression to evaluate the remaining b_i and B_i.

To determine the b_i and B_i, one makes a change of variable $X_i = b_i B_i, x_i = X_i^2 = \pm b_i^2$ and considers the linear system of equations in X_i [9]

$$P_1 = B = \sum_{i=1}^{m} X_i \tag{36}$$

$$P_2 = (B \wedge B) .B = 2 \sum_{i,j=1}^{m} X_i x_j \;\; (i \neq j) \tag{37}$$

$$P_3 = (B \wedge B \wedge B) . (B \wedge B) = 3!2! \sum_{i,j,k=1}^{m} X_i x_j \; x_k \;\; (i \neq j, j < k) \tag{38}$$

$$...... \tag{39}$$

$$P_m = (B \wedge B \wedge ...m \text{ factors}) . (B \wedge B ... (m-1) \text{ factors}) \tag{40}$$

$$= m! \, (m-1)! \sum_{i=1}^{m} x_1 x_2 ... x_{i-1} x_{i+1} ... x_m X_i . \tag{41}$$

The determinant Δ is the product

$$\Delta = \left\{ m! \left[(m-1)! \right]^2 \left[(m-2)! \right]^2 ...1 \right\} \prod_{i,j=1}^{m} (x_i - x_j) \;\; (i \neq j, i < j) . \tag{42}$$

If $\Delta \neq 0$, one obtains the bivectors X_i as a function of P_m and x_i. To determine the x_i, one writes the equations

$$S_1 = P_1 .P_1 = \sum_{i=1}^{m} x_i \tag{43}$$

$$S_2 = P_2 .P_1 = 2! \sum_{i,j=1}^{m} x_i x_j \;\; (i \neq j) \tag{44}$$

$$S_3 = P_3 .P_1 = (3!)^2 \sum_{i,j,k=1}^{m} x_i x_j x_k \;\; (i \neq j, j < k) \tag{45}$$

$$...... \tag{46}$$

$$S_m = P_m .P_1 = (m!)^2 \, (x_1 x_2 ... x_m) . \tag{47}$$

The solutions yield $x_i = \pm b_i^2$, thus one obtains b_i and B_i

$$b_i = \sqrt{|x_i|}, B_i = \frac{X_i}{b_i} . \tag{48}$$

If $\Delta = 0$, the equations (36-41) are not independent, the B bivector can nevertheless be decomposed in m mutually orthogonal simple planes but this decomposition is not unique.

4 Poincaré Group in n Dimensions (via Dual Hyperquaternions)

Much of physics being covariant with respect to the $4D$ Poincaré group, we provide here a hyperquaternion representation of the nD Poincaré groups in terms of dual hyperquaternions. Thereby one comes back to a $(n+1)D$ rotation which one can be decomposed canonically. The procedure is illustrated by a $5D$ case (for example a color image with 2 spatial and 3 color dimensions) which might be of interest in machine learning [10].

4.1 General Formalism

The Poincaré group of the pseudo-euclidean space associated with the Clifford algebra $C_{p,q}$ $(n = p + q)$ is constituted by the isometries of the metric

$$ds^2 = \left(dx_1^2 + ... + dx_p^2\right) - \left(dx_{p+1}^2 + ... + dx_{p+q}^2\right). \tag{49}$$

It includes the rotations $SO^+(p,q)$, translations and reflections (time or space-like). The reflections having already been dealt with above, we shall focus on the rotations and translations.

Consider a hyperquaternion algebra $\mathbb{H} \otimes \mathbb{H} ... \otimes \mathbb{H}$ (or a subalgebra thereof) with $n+1$ generators $e_1, e_2, ...e_n, e_{n+1}$ and let X be a dual vector such that

$$X = e_{n+1} + \varepsilon x \tag{50}$$

where x belongs to the vector space V_1 with $x = \sum_{i=1}^{n} e_i x_i$ $(x_i \in \mathbb{R})$ and $\varepsilon^2 = 0$ (ε commuting with e_i). An nD hyperbolic rotation in V_1 leaves the last variable unchanged. Hence,

$$X' = aXa_c = e_{n+1} + \varepsilon x' \tag{51}$$

with $x' = axa_c, x'x'_c = xx_c, aa_c = 1$. A translation in V_1 can be expressed as

$$X' = bXb_c \tag{52}$$

with

$$b = e^{\varepsilon e_{n+1}\frac{t}{2}} = 1 + \varepsilon e_{n+1}\frac{t}{2}, (t = \sum_{i=1}^{n} e_i t_i \ , t_i \in \mathbb{R}) \tag{53}$$

and $bb_c = 1$. Developing Eq. (52), one obtains, assuming $e_{n+1}^2 = -1$

$$X' = \left(1 + \varepsilon e_{n+1}\frac{t}{2}\right)(e_{n+1} + \varepsilon x)\left(1 - \varepsilon e_{n+1}\frac{t}{2}\right) \tag{54}$$

$$= e_{n+1} + \varepsilon x - \varepsilon e_{n+1} e_{n+1}\frac{t}{2} - \varepsilon e_{n+1} e_{n+1}\frac{t}{2} \tag{55}$$

$$= e_{n+1} + \varepsilon(x + t) \tag{56}$$

which is a translation on the variables $1...n$ (if $e_{n+1}^2 = 1$, one simply takes $b = e^{\varepsilon \frac{t}{2} e_{n+1}}$). A combination of an nD rotation and translation gives with $f = ab$ (or ba)

$$X' = fXf_c \quad \left(ff_c = 1,\ f \in C^+\right) \tag{57}$$

which can be viewed as a a particular $(n+1)\,D$ rotation. One thus obtains a hyperquaternion representation of the Poincaré groups, distinct from the matrix one. A canonical decomposition leads to simple dual planes as will be illustrated in the following example.

4.2 Example: 5D Poincaré Group

As application consider a $5D$-space (for example a $2D$ color image) imbedded in the $6D$ hyperquaternion algebra $\mathbb{H}\otimes\mathbb{H}\otimes\mathbb{H}$ having six generators (see Appendix)

$$e_1 = kI, e_2 = kJ, e_3 = kKl, e_4 = kKm, e_5 = kKn, e_6 = j \tag{58}$$

with the generic vector $X = e_6 + \varepsilon x \quad (x = \sum_{i=1}^{5} e_i x_i)$. The transformation $X' = fXf_c$ with

$$f = e^{\frac{\Phi_2}{2} Jl} e^{\varepsilon i(2I+Kn)} e^{\frac{\Phi_1}{2} I(m+n)} \tag{59}$$

$$= \left(2 + \sqrt{3}Jl\right) \left[1 + \varepsilon i\left(2I + Kn\right)\right] \left[\sqrt{3} + \sqrt{2}I\left(\frac{m}{\sqrt{2}} + \frac{n}{\sqrt{2}}\right)\right] \tag{60}$$

and $\tanh\frac{\Phi_1}{2} = \sqrt{\frac{2}{3}}\,(= b_1)$, $\tanh\frac{\Phi_2}{2} = \frac{\sqrt{3}}{2}\,(= b_2)$ is a $5D$-Poincaré transform. Applying the canonical decomposition presented above, one obtains

$$f = e^{\frac{\Phi_2}{2} B_2} e^{X_3} e^{\frac{\Phi_1}{2} B_1} \tag{61}$$

with the same values of Φ_1, Φ_2 as above and the following simple commuting orthogonal dual planes B_1, B_2, X_3

$$B_1 = \frac{1}{\sqrt{2}} I\left(m+n\right) + \varepsilon \frac{1}{\sqrt{2}} \left[\frac{\sqrt{3}}{2} K\left(m+n\right) - iJ\right] \tag{62}$$

$$B_2 = Jl + 2\varepsilon i\left(\frac{2}{\sqrt{3}} I - Kl\right) \tag{63}$$

$$X_3 = \frac{\varepsilon}{2} iK\left(-m+n\right). \tag{64}$$

with $(B_1)^2 = (B_2)^2 = 1, (X_3)^2 = 0$.

5 Conclusion

The paper has given a hyperquaternion representation of pseudo-euclidean rotations and the Poincaré groups in n dimensions, distinct from the matrix one. A canonical decomposition of these groups was introduced, as an extension of an euclidean formalism, within a hyperquaternion Clifford algebra framework and illustrated by a $5D$ example. Potential geometric applications include in particular, moving reference frames and machine learning.

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A Multivector structure of $\mathbb{H} \otimes \mathbb{H} \otimes \mathbb{H}$

$$
\begin{bmatrix}
1 & l = e_4e_5 & m = e_5e_3 & n = e_3e_4 \\
I = e_2e_3e_4e_5 & I\ l = e_3e_2 & I\ m = e_4e_2 & I\ n = e_5e_2 \\
J = e_3e_1e_4e_5 & J\ l = e_1e_3 & J\ m = e_1e_4 & J\ n = e_1e_5 \\
K = e_2e_1 & Kl = e_2e_1e_4e_5 & Km = e_1e_2e_3e_5 & Kn = e_2e_1e_3e_4
\end{bmatrix}
+ i \begin{bmatrix}
1 = e_1e_2e_3e_4e_5e_6 & l = e_2e_1e_3e_6 & m = e_2e_1e_4e_6 & n = e_2e_1e_5e_6 \\
I = e_6e_1 & I\ l = e_4e_1e_5e_6 & I\ m = e_5e_1e_3e_6 & I\ n = e_3e_1e_4e_6 \\
J = e_6e_2 & J\ l = e_4e_2e_5e_6 & J\ m = e_5e_2e_3e_6 & J\ n = e_3e_2e_4e_6 \\
K = e_3e_4e_5e_6 & Kl = e_6e_3 & Km = e_6e_4 & Kn = e_6e_5
\end{bmatrix}
$$

$$
+ j \begin{bmatrix}
1 = e_6 & l = e_4e_5e_6 & m = e_6e_5e_3 & n = e_3e_4e_6 \\
I = e_2e_3e_4e_5e_6 & I\ l = e_3e_2e_6 & I\ m = e_6e_4e_2 & I\ n = e_6e_5e_2 \\
J = e_4e_3e_5e_6e_1 & J\ l = e_1e_3e_6 & J\ m = e_1e_4e_6 & J\ n = e_1e_5e_6 \\
K = e_2e_1e_6 & Kl = e_2e_1e_4e_5e_6 & Km = e_1e_2e_3e_5e_6 & Kn = e_2e_1e_3e_4e_6
\end{bmatrix}
+ k \begin{bmatrix}
1 = e_2e_1e_3e_4e_5 & l = e_1e_2e_3 & m = e_1e_2e_4 & n = e_1e_2e_5 \\
I = e_1 & I\ l = e_1e_4e_5 & I\ m = e_3e_1e_5 & I\ n = e_1e_3e_4 \\
J = e_2 & J\ l = e_2e_4e_5 & J\ m = e_3e_2e_5 & J\ n = e_2e_3e_4 \\
K = e_4e_3e_5 & Kl = e_3 & Km = e_4 & Kn = e_5
\end{bmatrix}
$$

References

1. Ungar, A.: *Beyond Pseudo-Rotations in Pseudo-Euclidean Spaces*. Academic Press, London (2018)
2. Ferreira, M., Sommen, F.: *Complex Boosts: A Hermitian Clifford Algebra Approach*. Advances in Applied Clifford Algebras, 23: 339-362 (2013)
3. Lounesto, P.: *Clifford Algebras and Spinors*. Cambridge University Press, Cambridge (2001)
4. Crumeyrolle, A.: *Orthogonal and Symplectic Clifford Algebras: Spinor Structures*. Kluwer Academic Publishers, Dordrecht (1990)
5. Perez-Gracia, A., Thomas, F.: *On Cayley's Factorization of 4D Rotations and Applications*. Advances in Applied Clifford Algebras, 27:523-538 (2017)
6. Richard, A., Fuchs, L., Andres, E., Largeteau-Skapin, G.: *Decomposition of nD-rotations: Classification, properties and algorithm*. Graphical Models, Elsevier, 73 (6), pp. 346-353 (2011).
7. Girard, P.R., Clarysse, P., Pujol, R., A.,Goutte, R, Delachartre, P.: *Hyperquaternions: A New Tool for Physics*. Advances in Applied Clifford Algebras, 28:68 (2018)

8. Moore, C.L.E.: *Hyperquaternions.* Journal of Mathematics and Physics, vol. 1, pp. 63-77 (1922)
9. Moore, C.L.E.: *Rotations in Hyperspace.* Proceedings of the American Academy of Arts and Sciences, vol. 53, No. 8, pp. 651-694 (1918)
10. Xuanyu Zhu, Yi Xu, Hongteng Xu, Changjian Chen: *Quaternion Convolutional Neural Networks.* ECCV (2018)
11. Girard, P.R., Clarysse, P., Pujol, R., A., Wang, L, Delachartre, P.: *Differential Geometry Revisited by Biquaternion Clifford Algebra.* In J.-D. Boissonnat et al (Eds.): Curves and Surfaces. Springer (2015)
12. Girard, P.R.: *Quaternions, Clifford Algebras and Relativistic Physics.* Birkhäuser, Basel (2007)
13. Girard, P.R.: *Algèbre de Clifford et Physique relativiste* . PPUR, Lausanne (2004)
14. Girard, P.R.: *Einstein's equations and Clifford algebra.* Advances in Applied Clifford Algebras, vol. 9, 2, pp. 225-230 (1999)
15. Girard, P.R.: *The quaternion group and modern physics.* Eur. J. Phys., vol. 5, pp. 25-32 (1984)
16. Casanova, G.: *L'algèbre vectorielle.* PUF, Paris (1976)
17. Lipschitz, R.: *Principes d'un calcul algébrique qui contient comme espèces particulières le calcul des quantités imaginaires et des quaternions.* C.R. Acad. Sci. Paris, vol. 91, pp. 619-621, 660-664 (1880)