# Phase Extraction Of The Analytic Video Signal In Clifford Algebra

Liang Wang, Patrick R. Girard, Patrick Clarysse and Philippe Delachartre

**Abstract** The paper presents an algebraic framework for $(2D+t)$ video analytic signals and a numerical implementation thereof using Clifford biquaternions and Clifford Fourier transforms. Though the basic concepts of Clifford Fourier transforms are well-known, an implementation of analytic video sequences using multiquaternion algebras does not seem to have been realized so far. A biquaternion algebraic framework is developed to express Clifford Fourier transforms and $(2D+t)$ video analytic signals in standard and polar form constituted by a scalar, a pseudo-scalar and six phases. The phase extraction procedure is provided. Then a numerical implementation using discrete fast Fourier transforms of an analytic multiquaternion video signal is proposed. When considering a progressive plane wave, the proposed numerical estimation retrieves correctly the six phases and the linear relationships with the time, horizontal position and vertical position of video signal.

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# 1 Introduction

During the last decades, based on Clifford algebras, new algebraic structures [16] and complex analytic signal [10, 8, 4, 15] have been developed. Unser et al [15] set the mathematical foundations of a monogenic wavelet transform that gives access to the local orientation, amplitude and phase information of a 2-D signal. Brackx et al [1] defined a Clifford Fourier transforms in the framework of Clifford analysis, that is shown to be a refinement of the classical multi-dimensional Fourier transform. In [10], Sangwine and Le Bihan introduces some properties of the hyper-analytic signal and its complex envelope, which are the extension of an analytic signal concept. As a mathematical tool, the quaternion-based methods have been applied to image analysis and signal processing, such as motion estimation and tracking [9, 11], motion detection [14] and disparity image [2]. In addition, quaternion-based methods are also useful for edge detection of color image thanks to its multicomponent property. Because each quaternion component can be associated with an RGB component [12].

Generally, phase information can be obtained from quaternion-based method. Several works have shown previously the advantage of using phase information to analyse image and image sequences. In [3], Bülow found that a component of the quaternionic phase is sensitive to change of a structure like a flaw in a textile, which could be localized by thresholding a phase. In 2006, Ben Witten [17] proposed an image disparity estimation method, based on a hypercomplex phase-based technique enabling to find differences between subtly varying images. Woo [18] used both the local phase information and the intensity information for ultrasound image registration. Felsberg [5] used the monogenic phase for optical flow estimation. He indicated that: (1) there is strong evidence that the human visual system makes use of local phase; (2) phase-based processing is to a large extend invariant to changes of lighting conditions; (3) the reconstruction of an image from phase information is much better than that from amplitude information.

Though the basic concepts of Clifford algebras and Clifford Fourier transforms are known, an implementation of analytic video sequences using multiquaternion algebras does not seem to have been realized so far. In this paper, $2D+t$ analytic video signal is defined. Therefore, it has three components: time, vertical position and horizontal position. With the objective to process an entire video signal at once, a biquaternion algebraic framework is developed to express Clifford Fourier transforms of $2D+t$ video analytic signals in standard and polar forms constituted by a scalar, a pseudo-scalar and six phases.

Finally, a numerical implementation using discrete fast Fourier transforms to obtain an analytic multiquaternion video signal is provided. On the specific case of a progressive plane moving wave, approximate linear relationships between phases $(\varphi_1, \varphi_2, \varphi_3)$ and the three components of video signal $(t, x, y)$ are found.

# 2 Clifford Biquaternion 2D+t Analytic Signal: Implementation

## 2.1 Clifford Biquaternion Algebra

In $2D+t$ dimensions, we shall take as Clifford algebra Clifford biquaternions having as generators ($e_1 = \varepsilon i, e_2 = \varepsilon j; e_3 = \varepsilon k, \varepsilon = i'I, \varepsilon^2 = 1, e_i^2 = -1$) with $i'$ designating the usual complex imaginary ($i'^2 = -1$) and where the tensor product $I = 1 \otimes i$ ($I^2 = -1$) commutes with $i = i \otimes 1, j = j \otimes 1, k = k \otimes 1$; $e_1$ corresponds to the time axis $x_1$, $e_2$ and $e_3$ correspond respectively to the $x_2$ and $x_3$ axes. The full algebra contains the elements:

$$\begin{bmatrix} 1 & i = e_2e_3 & j = e_3e_1 & k = e_1e_2 \\ \varepsilon = i'I = -e_1e_2e_3 & e_1 = \varepsilon i & e_2 = \varepsilon j & e_3 = \varepsilon k \end{bmatrix}. \tag{1}$$

Hence, a general element of the algebra can be expressed as a Clifford biquaternion

$$\begin{aligned} A &= p + \varepsilon q \\ &= (p_0 + ip_1 + jp_2 + kp_3) + \varepsilon(q_0 + iq_1 + jq_2 + kq_3) \\ &= [p_0, p_1, p_2, p_3] + \varepsilon\,[q_0, q_1, q_2, q_3]. \end{aligned} \tag{2}$$

The conjugate of $A$ is

$$\begin{aligned} A_c &= p_c + \varepsilon q_c \\ &= [p_0, -p_1, -p_2, -p_3] + \varepsilon\,[q_0, -q_1, -q_2, -q_3], \end{aligned} \tag{3}$$

where $p_c$, $q_c$ are respectively the quaternion conjugates of $p$ and $q$. The complex conjugate of $A$ is

$$\overline{A} = p - \varepsilon q. \tag{4}$$

## 2.2 Clifford Fourier Transform

Given a function $f(x)$ having its value in the Clifford algebra with $x = (x_1, x_2, x_3)$, and let $F(u)$ with $u = (u_1, u_2, u_3)$ denote the Clifford Fourier transform:

$$F(u) = \int_{R^3} f(x) e^{-\varepsilon i 2\pi u_1 x_1} e^{-\varepsilon j 2\pi u_2 x_2} e^{-\varepsilon k 2\pi u_3 x_3} dx_1 dx_2 dx_3. \tag{5}$$

The inverse Clifford Fourier transform is given by:

$$f(x) = \int_{R^3} F(u) e^{\varepsilon k 2\pi u_3 x_3} e^{\varepsilon j 2\pi u_2 x_2} e^{\varepsilon i 2\pi u_1 x_1} du_1 du_2 du_3. \tag{6}$$

In this paper, the discrete calculation of Clifford Fourier transform (CFT) is based on the standard FFT algorithm. It is simply a discrete form of the continuous formula. To compute the direct Clifford Fourier transform, one proceeds in cascade integrating first with respect to $x_1$ using a standard FFT. A second FFT (integration with respect to $x_2$) is then applied on each real component of the previous complex number. Then, a third FFT (integration on $x_3$) is applied on each of the resulting real components. Finally, all the components are properly displayed as a Clifford biquaternion. For the inverse Clifford Fourier transform, one proceeds in the same way on each real component of the Clifford biquaternion using an IFFT and reversing the order of integration. Due to the symmetries of the Clifford Fourier transform of a scalar function $f(x_1, x_2, x_3)$, only one orthant of the Fourier space is necessary to obtain the entire Fourier space.

## 2.3 Polar Form Of The Analytic Signal

In polar form, one can obtain the module and the phase of an analytic signal. In $2D+t$ analytic video signal, there are a scalar and a pseudo-scalar and six phases. An algorithm to calculate the polar form of the $2D+t$ analytic video signal is illustrated in this section.

The analytic Clifford Fourier transform is defined by:

$$F_A(u) = [1 + sign(u_1)]\,[1 + sign(u_2)]\,[1 + sign(u_3)]\,F(u). \tag{7}$$

and the analytic signal by:

$$f_A(x) = \int_{R^3} F_A(u) e^{\varepsilon k 2\pi u_3 x_3} e^{\varepsilon j 2\pi u_2 x_2} e^{\varepsilon i 2\pi u_1 x_1} du_1 du_2 du_3. \tag{8}$$

In order to obtain the polar form of the analytic signal, a scalar, a pseudo-scalar and a unit Clifford biquaternion will be used to represent the analytic signal.

The unit Clifford biquaternion, that is called $a$ in the following part, contains six phases of an analytic signal. Firstly, the unit Clifford biquaternion $a$ is decomposed to two other unit Clifford biquaternions, which are called $b$ and $r$. Next, by representing $b$ and $r$ in polar form, three phases are defined as $\varphi_1, \varphi_2, \varphi_3$ for the unit Clifford biquaternion $b$ and three phases are defined as $\theta_1, \theta_2, \theta_3$ for the unit Clifford biquaternion $r$. Finally, the six phases $\varphi_1, \varphi_2, \varphi_3, \theta_1, \theta_2, \theta_3$ of the analytic signal are obtained.

### 2.3.1 Calculation Of The Unit Clifford Biquaternion a

The analytic signal being a Clifford biquaternion, it can be represented as $A = \lambda a$, where the Clifford biquaternion $A$ represents the analytic signal $f_A$, and $\lambda = \alpha + \varepsilon\beta$ is constituted by a scalar and a pseudo-scalar and where $a$ is a unit Clifford biquaternion ($aa_c = 1$). Writing, $AA_c = \lambda^2 = g_1 + \varepsilon g_2$, one finds

$$\alpha = \sqrt{\frac{g_1 + \sqrt{g_1^2 - g_2^2}}{2}}, \quad \beta = \sqrt{\frac{g_1 - \sqrt{g_1^2 - g_2^2}}{2}}. \tag{9}$$

(if $\alpha = \beta = 0$, one adopts the choice $a = 1$, in order to have a unit Clifford biquaternion $a$ in all cases). Then one has:

$$a = \lambda^{-1} A = (\alpha + \varepsilon\beta)^{-1} A = [\frac{\alpha - \varepsilon\beta}{\alpha^2 - \beta^2}]A. \tag{10}$$

#### 2.3.2 Calculation Of The Unit Clifford Biquaternion b and r

The unit Clifford biquaternion $a$ can be decomposed as $a = rb$, with $b = b_1 + \varepsilon\,(ib_2 + jb_3 + kb_4)$ being a unit Clifford biquaternion such that $\overline{b_c} = b$ and where $r = r_1 + ir_2 + jr_3 + kr_4$ with $rr_c = 1$. Using a procedure similar to that used in special relativity [6, p. 82], one has

$$b = \frac{1 + d}{\sqrt{2 + d + d_c}}. \tag{11}$$

with $d = (\overline{a_c})a$. The unit Clifford biquaternion $r$ is obtained as $r = ab_c$.

#### 2.3.3 Calculation Of The Six Phases Of The Analytic Video Signal

Both, $b$ and $r$ can be put into a polar form, respectively

$$\begin{aligned} b \ &= e^{\varepsilon j\varphi_2}\left[e^{\varepsilon k\varphi_3}\left(e^{\varepsilon i\varphi_1}\right)e^{\varepsilon k\varphi_3}\right]e^{\varepsilon j\varphi_2} \\ &= \cos\varphi_1\cos 2\varphi_2\cos 2\varphi_3 + \varepsilon(i\sin\varphi_1 \\ &\quad + j\cos\varphi_1\cos 2\varphi_3\sin 2\varphi_2 + k\cos\varphi_1\sin 2\varphi_3), \end{aligned} \tag{12}$$

with $\overline{b_c} = b$ and $r = e^{i\theta_1}e^{k\theta_3}e^{j\theta_2}$. The phases of $b$ are extracted according to the rules:

$$\varphi_1 = \arcsin b\,(2,2)\,, \tag{13}$$

where $b\,(2,2)$ means the second component of the second quaternion of $b$. If $\cos\varphi_1 \neq 0$,

$$\varphi_3 = \frac{1}{2}\arcsin\left(\frac{b\,(2,4)}{\cos\varphi_1}\right), \tag{14}$$

if $\cos\varphi_1 \neq 0$ and $\cos 2\varphi_3 \neq 0$

$$\varphi_2 = \frac{1}{2}\arcsin\left(\frac{b\,(2,3)}{\cos\varphi_1\cos 2\varphi_3}\right), \tag{15}$$

The phases of $r = e^{i\theta_1} e^{k\theta_3} e^{j\theta_2}$ are extracted according to the procedure presented for the $2D$ analytic signal by Sommer [13, p. 194] (except that eventual phase shifts are reported on $\theta_3$ rather than on $\theta_1$). Within the $2D+t$ Clifford biquaternion framework, the procedure of extracting the triplet of phases $(\theta_1, \theta_2, \theta_3)$ goes as follows:

From the relations

$$d_1 = r * [K_2(r)]_c = [\cos 2\theta_1 \cos 2\theta_3, \sin 2\theta_1 \cos 2\theta_3, 0, \sin 2\theta_3], \tag{16}$$

$$d_2 = [K_1(r)]_c * r = [\cos 2\theta_2 \cos 2\theta_3, 0, \cos 2\theta_3 \sin 2\theta_2, \sin 2\theta_3], \tag{17}$$

one obtains

$$\theta_3 = \frac{Arc\sin d_1(4)}{2}, \tag{18}$$

where the operator $*$ represents the quaternion multiplication. $K_1, K_2, K_{12}$ are the involutions defined in [7], and $d_1(4)$ means the fourth component of $d_1$. If $\cos 2\theta_3 \neq 0$, one has

$$\theta_1 = \frac{Arg[d_1(1) + i d_1(2)]}{2}, \theta_2 = \frac{Arg[d_2(1) + i d_2(3)]}{2}. \tag{19}$$

If $\cos 2\theta_3 = 0$, one has an undeterminacy and only $(\theta_1 \pm \theta_2)$ can be determined from the relation $d_3 = [K_{12}(r)]_c * r = [\cos 2(\theta_1 \mp \theta_2), 0, \mp \sin 2(\theta_1 \mp \theta_2), 0]$. Adopting the choice $\theta_1 = 0$, one has

$$\theta_2 = \frac{Arg[d_3(1) + i d_3(3)]}{2}. \tag{20}$$

The particular cases of the six phases are treated in [7], and finally, the analytic signal is characterized by a scalar, a pseudo-scalar and the six phases given in this part.

## 3 Application To A Progressive Plane Wave

In order to put into application, a progressive plane moving wave is created by a cosine function. Then its analytic video signal and the polar form of the analytic video signal are calculated.

### 3.1 Algebraic Result

One considers the progressive plane moving wave given by:

$$f(x_0, x_1, x_2) = \cos(\omega_0 x_0 + \omega_1 x_1 + \omega_2 x_2), \tag{21}$$

where $\omega_0, \omega_1$ and $\omega_2$ are the pulsations of the plane wave. $x_0$ represents the time axis $t$, $x_1$ represents the horizontal axis $x$ and $x_2$ represents the vertical axis $y$ of a frame in an image sequence. The analytic signal is given by:

$$
\begin{aligned}
f_A(x_0,x_1,x_2) = & \begin{bmatrix} \cos(\omega_0 x_0 + \omega_1 x_1 + \omega_2 x_2), \\ -\cos(\omega_0 x_0 - \omega_1 x_1 + \omega_2 x_2), \\ \cos(\omega_0 x_0 + \omega_1 x_1 - \omega_2 x_2), \\ -\cos(\omega_0 x_0 - \omega_1 x_1 - \omega_2 x_2) \end{bmatrix} \\
& +\varepsilon \begin{bmatrix} \sin(\omega_0 x_0 - \omega_1 x_1 + \omega_2 x_2), \\ \sin(\omega_0 x_0 + \omega_1 x_1 + \omega_2 x_2), \\ -\sin(\omega_0 x_0 - \omega_1 x_1 - \omega_2 x_2), \\ -\sin(\omega_0 x_0 + \omega_1 x_1 - \omega_2 x_2), \end{bmatrix}, \qquad (22)
\end{aligned}
$$

then the unit Clifford biquaternion $a = \frac{f_A}{2}$, with $aa_c = 1$. Using $d = (\overline{a_c})a$ and equation 11, one has

$$
\begin{aligned}
b = & \left[\frac{1+\cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}{\alpha}, 0, 0, 0\right] \\
& +\varepsilon \left[0, \frac{\cos 2\omega_1 x_1 \cos 2\omega_2 x_2 \sin 2\omega_0 x_0}{\alpha}, \frac{\cos 2\omega_2 x_2 \sin 2\omega_1 x_1}{\alpha}, \frac{\sin 2\omega_2 x_2}{\alpha}\right] \qquad (23)
\end{aligned}
$$

with $\alpha = \sqrt{2+2\cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}$. Then by equations 13, 14 and 15, one has

$$
\varphi_1 = \arcsin\left(\frac{\cos 2\omega_1 x_1 \cos 2\omega_2 x_2 \sin 2\omega_0 x_0}{\sqrt{2+2\cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}}\right), \qquad (24)
$$

$$
\varphi_2 = \frac{1}{2\cos\varphi_1 \cos 2\varphi_3} \arcsin\left(\frac{\cos 2\omega_2 x_2 \sin 2\omega_1 x_1}{\sqrt{2+2\cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}}\right), \qquad (25)
$$

$$
\varphi_3 = \frac{1}{2\cos\varphi_1} \arcsin\left(\frac{\sin 2\omega_2 x_2}{\sqrt{2+2\cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}}\right), \qquad (26)
$$

if all $\omega_0 x_0 \ll 1, \omega_1 x_1 \ll 1, \omega_2 x_2 \ll 1$, one has $\sin\varphi_1 \approx \frac{\sin 2\omega_0 x_0}{2}, \sin 2\varphi_2 \approx \frac{\sin 2\omega_1 x_1}{2}$, and $\sin 2\varphi_3 \approx \frac{\sin 2\omega_2 x_2}{2}$. Hence,

$$
\varphi_1 \approx \omega_0 x_0, \quad \varphi_2 \approx \frac{\omega_1 x_1}{2}, \quad \varphi_3 \approx \frac{\omega_2 x_2}{2}. \qquad (27)
$$

Phases $(\theta_1, \theta_2, \theta_3)$ are given by equations 28, 29 and 30:

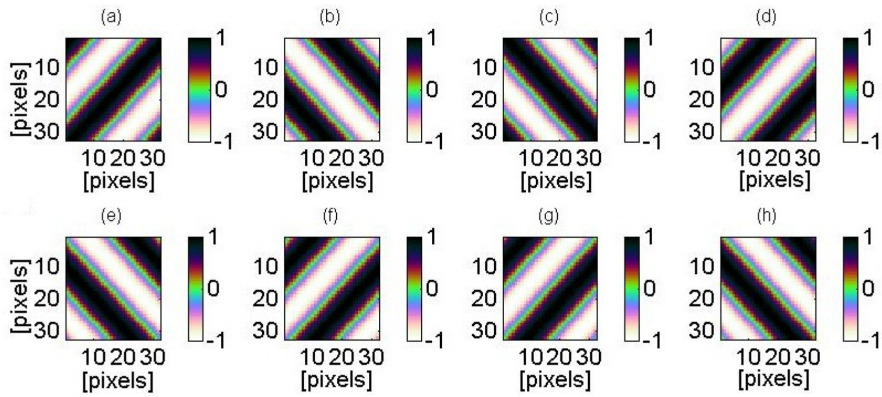


**Fig. 1** The eight components of the analytic video signal $f_A = (a,b,c,d) + \varepsilon(e,f,g,h)$ at frame $x_0 = 31$.

$$\sin 2\theta_3 = \frac{\sin 2\omega_1 x_1 \sin 2\omega_2 x_2}{1 + \cos 2\omega_0 x_0 \cos 2\omega_1 x_1 \cos 2\omega_2 x_2}, \tag{28}$$

$$\tan 2\theta_1 = \frac{\cos 2\omega_1 x_1 + \cos 2\omega_0 x_0 \cos 2\omega_2 x_2}{\cos 2\omega_2 x_2 \sin 2\omega_0 x_0 \sin 2\omega_1 x_1}, \tag{29}$$

$$\tan 2\theta_2 = \frac{\cos(2\omega_0 x_0 - 2\omega_1 x_1) + \cos 2(\omega_0 x_0 + \omega_1 x_1) + 2\cos 2\omega_2 x_2}{\cos 2\omega_1 x_1 \sin 2\omega_0 x_0 \sin 2\omega_2 x_2}. \tag{30}$$

In addition, when $\omega_0 x_0 \ll 1, \omega_1 x_1 \ll 1, \omega_2 x_2 \ll 1$, one has the phases $\theta_1 = -\frac{\pi}{4}, \theta_2 = \frac{\pi}{4}, \theta_3 = 0$.

## 3.2 Numerical Approximation

The numerical approximation is calculated via matlab. For the progressive plane wave video signal $f(x_0,x_1,x_2) = \cos(\omega_0 x_0 + \omega_1 x_1 + \omega_2 x_2)$, one has the analytic video signal $f_A$ as in Fig. 1,with normalized pulsation value of $\omega_0 = \omega_1 = \omega_2 = \frac{\pi}{16}$.

In polar form, the eight components $(\alpha, \theta_1, \theta_2, \theta_3, \beta, \varphi_1, \varphi_2, \varphi_3)$ of this analytic video signal are represented in Fig. 2. Here $\alpha$ and $\beta$ are given by equations 9.

From equation 27, there are approximate linear relationships between phases $\varphi_1, \varphi_2, \varphi_3$ and $x_0, x_1, x_2$ around the origin point of the axes $x_0, x_1, x_2$. Fig. 3$(a),(b),(c)$ shows the computed profiles of $\varphi_1, \varphi_2, \varphi_3$ respectively at $(x_1,x_2), (x_0,x_2)$ and $(x_0,x_1)$ fixed. The slopes at the origin give the $\omega_0, \omega_1$ and $\omega_2$ values which should be equal to $\frac{\pi}{16} = 0.1963$. Thanks to the numerical approximation, the estimated values of $\omega_0, \omega_1$ and $\omega_2$ are 0.1736, 0.2068 and 0.1952 respectively. Hence, one can find the displacements of $x_1$ and $x_2$ from the phases near the origin. Fig. 4 shows the computed profiles of $\theta_1, \theta_2, \theta_3$ respectively at $(x_1,x_2), (x_0,x_2)$ and $(x_0,x_1)$ fixed. The values at the origin correspond to the expected values, i.e. $-\frac{\pi}{4}, \frac{\pi}{4}$ and 0, respectively.

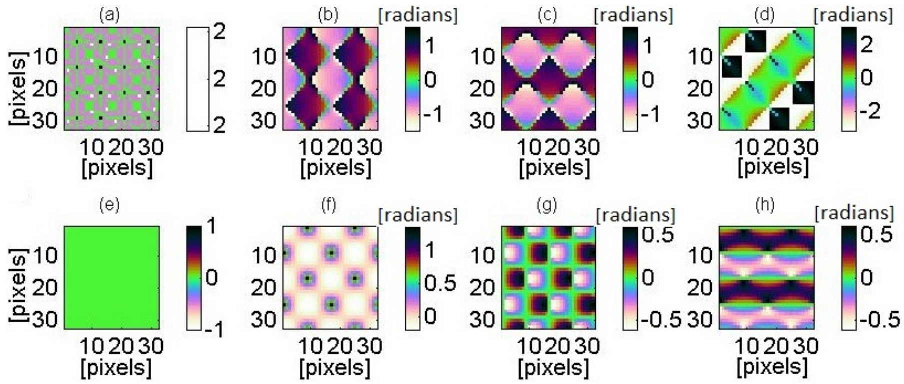


**Fig. 2** Polar form of the analytic video signal at the frame $x_0 = 31$. $(a)$ is the scalar, $(b,c,d)$ are the three phases $(\theta_1, \theta_2, \theta_3)$; (e) is the pseudo-scalar and $(f,g,h)$ are the three phases $(\varphi_1, \varphi_2, \varphi_3)$.

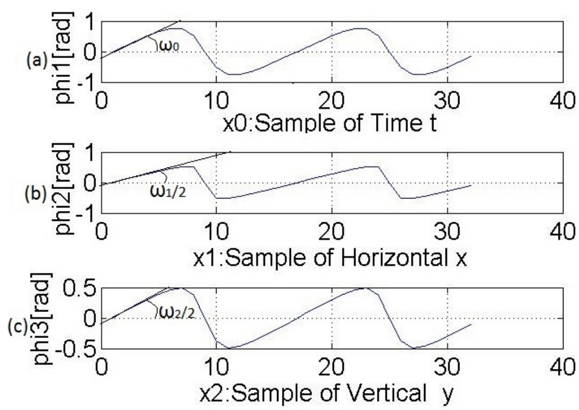


**Fig. 3** $(a)$ Angle $\varphi_1$ along time axis $x_0$. Its slope near origin is $\omega_0$. $(b)$ Angle $\varphi_2$ along horizontal axis $x_1$. Its slope near origin is $\frac{\omega_1}{2}$. $(c)$ Angle $\varphi_3$ along vertical axis $x_2$, Its slope near origin is $\frac{\omega_2}{2}$.

Moreover, the phase shift observed on $\theta_3$ corresponds to the phase shift introduced to guarantee the proper sign of $r$ (see section 2.3.3).

## 4 Conclusion

The paper has presented a concrete algebraic framework, i.e. Clifford's biquaternions for the expression of Clifford Fourier transforms and $2D+t$ analytic signals. Then, we have shown how to put the analytic signal into a polar form constituted by a scalar, a pseudo-scalar and six phases. Finally, using discrete fast Fourier transforms we have implemented numerically the Clifford biquaternion Fourier transform, the analytic Fourier transform and the analytic signal both in standard and polar form. With the example of a progressive plane wave, we have shown that our implementation is able to correctly recover the phases and the linear relation vs the

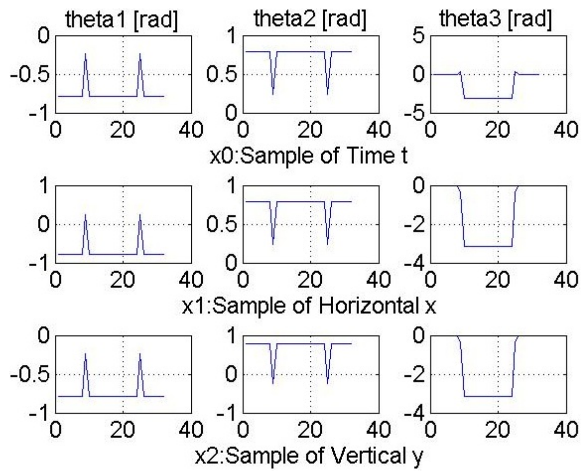


**Fig. 4** First column: phase $\theta_1$ corresponding to three axes. All the values near origin are $-\frac{\pi}{4}$; Second column: phase $\theta_2$ corresponding to three axes. All the values near origin are $\frac{\pi}{4}$; Third column: phase $\theta_3$ corresponding to three axes. All the values near origin are 0.

axes. This opens the way to the exploitation of the phase information. In this paper, a general Clifford Fourier transform processing for three dimension signal is given. However, some properties of the "t" dimension is interested to be studied. Our next objective will be to extract physical parameters and to find the relation between these physical parameters and the input images.

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