Technical Report
Derivatives Calculation for Constrained Bundle Adjustment

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1 Constrained Bundle Adjustment - reminder

In this section, we shortly remind the constrained bundle adjustment formulation with a first constraint function g_0. We have a set of M camera and N points. Observation on camera j of the point i is noted $\boldsymbol{x}_j^i$ and camera parameters are represented as vectors $\boldsymbol{\alpha}_j = (\boldsymbol{v}_j, \boldsymbol{t}_j)^T$. $\boldsymbol{v}_j$ and $\boldsymbol{t}_j$ are respectively the rotation and translation vectors of the camera j. $\hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)$ is the projection of the estimated point $\boldsymbol{X}^i$ on camera j with estimated parameters $\boldsymbol{\alpha}_j$.The optimization function in our model is written:

$$\arg\min_{\boldsymbol{\alpha}_j, \boldsymbol{X}^i} \sum_{i,j} \left[\boldsymbol{x}_j^i - \hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)\right]^2 + \mu_0 \sum_i \left[g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)\right]^2, \tag{1}$$

where μ_0 is the first penalty parameter,

$$\hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i) = \frac{1}{c_j^i} \boldsymbol{K}_j \boldsymbol{l}_j^i, \tag{2}$$

where $\boldsymbol{K}_j$ is the 2×3 matrix of the intrinsic camera parameters, considered as known after system calibration, and

$$\boldsymbol{l}_j^i = (a_j^i, b_j^i, c_j^i)^T = \left[\boldsymbol{R}_j^T, -\boldsymbol{R}_j^T \boldsymbol{t}_j\right] \begin{pmatrix} \boldsymbol{X}^i \\ 1 \end{pmatrix} \tag{3}$$

with $\boldsymbol{R}_j$ the rotation matrix formed using Euler-Rodrigues formula [1]

$$\boldsymbol{R}_j = \mathrm{Id} + \sin\theta [\bar{\boldsymbol{v}}_j]_\times + (1 - \cos\theta)[\bar{\boldsymbol{v}}_j]_\times^2,$$

where θ is the rotation angle around the unit vector $\bar{\boldsymbol{v}}_j = (\bar{v}_{j,1}, \bar{v}_{j,2}, \bar{v}_{j,3})$, such that $\boldsymbol{v}_j = \theta \bar{\boldsymbol{v}}_j$, and $[\bar{\boldsymbol{v}}_j]_\times$ is the cross product matrix defined as

$$[\bar{\boldsymbol{v}}_j]_\times = \begin{pmatrix} 0 & -\bar{v}_{j,3} & \bar{v}_{j,2} \\ \bar{v}_{j,3} & 0 & -\bar{v}_{j,1} \\ -\bar{v}_{j,2} & \bar{v}_{j,1} & 0 \end{pmatrix} \tag{4}$$

And $g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i) = \max(0, g_0(\boldsymbol{\alpha}_r, \boldsymbol{X}^i))$, with in the simple example

$$g_0(\boldsymbol{\alpha}_r, \boldsymbol{X}^i) = \sqrt{(\tilde{x}^i - x_{\text{init}}^i)^2 + (\tilde{y}^i - y_{\text{init}}^i)^2} - a \tag{5}$$

Registration:

$$\tilde{\boldsymbol{X}}^i = \boldsymbol{P} \begin{bmatrix} \boldsymbol{R}_r^T & -\boldsymbol{R}_r^T \boldsymbol{t}_r \\ 0 & 1 \end{bmatrix} \begin{pmatrix} \boldsymbol{X}^i \\ 1 \end{pmatrix}. \tag{6}$$

2 Derivatives

Now we will calculate the non-zero elements of the Jacobian:

$$\frac{\partial(\boldsymbol{x}_j^i - \hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i))^2}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)} = -2\,\mathrm{diag}(\boldsymbol{x}_j^i - \hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i))\frac{\partial \hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}, \tag{7}$$

where

$$\frac{\partial \hat{\boldsymbol{x}}(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)} = \boldsymbol{K}_j(\frac{1}{c_j^i}\frac{\partial \boldsymbol{l}_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)} - \frac{1}{(c_j^i)^2}\boldsymbol{l}_j^i\frac{\partial c_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}), \tag{8}$$

where

$$\frac{\partial \boldsymbol{l}_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)} = (\frac{\partial a_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}, \frac{\partial b_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}, \frac{\partial c_j^i}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)})^T = \frac{\partial \boldsymbol{T}_j}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)}\begin{pmatrix}\boldsymbol{X}^i \\ 1\end{pmatrix} + \boldsymbol{T}_j\frac{\partial(\boldsymbol{X}^i, 1)^T}{\partial(\boldsymbol{\alpha}_j, \boldsymbol{X}^i)} \tag{9}$$

where $\boldsymbol{T}_j = \left[\boldsymbol{R}_j^T, -\boldsymbol{R}_j^T\boldsymbol{t}_j\right]$.

First regarding the points parameters, we have $\frac{\partial \boldsymbol{T}_j}{\partial \boldsymbol{X}^i} = 0$, thus

$$\frac{\partial \boldsymbol{l}_j^i}{\partial \boldsymbol{X}^i} = \boldsymbol{R}_j^T \tag{10}$$

Now, regarding the camera parameters, since $\boldsymbol{X}^i$ does not depend on the camera parameters, $\frac{\partial(\boldsymbol{X}^i,1)^T}{\partial \boldsymbol{\alpha}_j} = 0$, and

$$\frac{\partial \boldsymbol{l}_j^i}{\partial \boldsymbol{\alpha}_j} = \frac{\partial \boldsymbol{T}_j}{\partial \boldsymbol{\alpha}_j}\begin{pmatrix}\boldsymbol{X}^i \\ 1\end{pmatrix}, \tag{11}$$

where

$$\frac{\partial \boldsymbol{T}_j}{\partial \boldsymbol{\alpha}_j} = \left[\frac{\partial \boldsymbol{R}_j}{\partial \boldsymbol{\alpha}_j}^T, -\frac{\partial \boldsymbol{R}_j}{\partial \boldsymbol{\alpha}_j}^T\boldsymbol{t}_j - \boldsymbol{R}_j^T\frac{\partial \boldsymbol{t}_j}{\partial \boldsymbol{\alpha}_j}\right] \tag{12}$$

with

$$\frac{\partial \boldsymbol{t}_j}{\partial \boldsymbol{\alpha}_j} = [0_3, I_3]\,, \frac{\partial \boldsymbol{R}_j^T}{\partial \boldsymbol{\alpha}_j} = \left[\frac{\partial \boldsymbol{R}_j^T}{\partial v_{j,1}}, \frac{\partial \boldsymbol{R}_j^T}{\partial v_{j,2}}, \frac{\partial \boldsymbol{R}_j^T}{\partial v_{j,3}}, 0_3\right], \tag{13}$$

and $\forall p \in (1, 2, 3)$

$$\frac{\partial \boldsymbol{R}_j}{\partial v_{j,p}} = \cos\theta\bar{v}_{j,p}[\bar{\boldsymbol{v}}_j]_\times + \sin\theta\bar{v}_{j,p}[\bar{\boldsymbol{v}}_j]_\times^2 + \frac{\sin\theta}{\theta}[\boldsymbol{e}_p - \bar{v}_{j,p}\bar{\boldsymbol{v}}_j]_\times + \frac{1-\cos\theta}{\theta}(\boldsymbol{e}_p\bar{\boldsymbol{v}}_j^T + \bar{\boldsymbol{v}}_j\boldsymbol{e}_p^T - 2\bar{v}_{j,p}\bar{\boldsymbol{v}}_j\bar{\boldsymbol{v}}_j^T) \tag{14}$$

as demonstrated in [1], with (e1, e2, e3) the standard basis in $\mathbb{R}^3$.

Then, for the constraint formulation in (1)

$$\frac{\partial(g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i))^2}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} = 2g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)\frac{\partial g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}, \tag{15}$$

with

$$\frac{\partial g_0^+(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} = \begin{cases} 0 & \text{if } g_0(\boldsymbol{\alpha}_r, \boldsymbol{X}^i) \leq 0 \\ \dfrac{\partial g_0(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} & \text{otherwise} \end{cases} \tag{16}$$

In the particular case of (5),

$$\frac{\partial g_0(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} = \frac{1}{2}[(\tilde{x}^i - x_{\text{init}}^i)^2 + (\tilde{y}^i - y_{\text{init}}^i)^2]^{-1/2} \times [2(\tilde{x}^i - x_{\text{init}}^i)\frac{\partial \tilde{x}^i}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} + 2(\tilde{y}^i - y_{\text{init}}^i)\frac{\partial \tilde{y}^i}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}], \tag{17}$$

with $\tilde{\boldsymbol{X}}^i = (\tilde{x}^i, \tilde{y}^i, \tilde{z}^i)$ defined in (6), and

$$\frac{\partial \tilde{\boldsymbol{X}}^i}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)} = P\left[\frac{\partial \tilde{\boldsymbol{T}}^k}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}\begin{pmatrix}\boldsymbol{X}^i \\ 1\end{pmatrix} + \tilde{\boldsymbol{T}}^k\frac{\partial(\boldsymbol{X}^i, 1)^T}{\partial(\boldsymbol{\alpha}_r, \boldsymbol{X}^i)}\right]. \tag{18}$$

where $\tilde{\boldsymbol{T}}_j = \begin{bmatrix} \boldsymbol{R}_r^T & -\boldsymbol{R}_r^T \boldsymbol{t}_r \\ 0 & 1 \end{bmatrix}$.

Finally,

$$\frac{\partial \tilde{\boldsymbol{X}}^i}{\partial \boldsymbol{X}^i} = \boldsymbol{P} \begin{bmatrix} \boldsymbol{R}_r^T \\ 0 \end{bmatrix}, \tag{19}$$

and

$$\begin{cases} \dfrac{\partial \tilde{\boldsymbol{X}}^i}{\partial \boldsymbol{\alpha}_j} = 0 & \text{if } j \neq k \\ \dfrac{\partial \tilde{\boldsymbol{X}}^i}{\partial \boldsymbol{\alpha}_r} = \boldsymbol{P} \begin{bmatrix} \dfrac{\partial \boldsymbol{T}^k}{\partial \boldsymbol{\alpha}_r} \\ 0 \end{bmatrix} \begin{pmatrix} \boldsymbol{X}^i \\ 1 \end{pmatrix} & \end{cases}, \tag{20}$$

where $\dfrac{\partial \boldsymbol{T}^k}{\partial \boldsymbol{\alpha}_r}$ is calculated in (12) to (14).

References

[1] G. Gallego and A. Yezzi, "A compact formula for the derivative of a 3-d rotation in exponential coordinates," *Journal of Mathematical Imaging and Vision*, vol. 51, pp. 378–384, Mar. 2015.