Tdsi AR
26/01/2024
Durée : 60 minutes (including MCQ)

NOM : ____________________
Prénom : ____________________

This exercice has 7 pages. Fill you first name and last name on the top of the first page; put your initials on the next pages.

Only lesson notes are allowed.

Goal of the exercice

The goal of this exercise is to recover a 2D image from its Radon transform. We will derive an exact reconstruction formula, known as the *filtered back-propagation formula.*

Reminders

- The Radon transform $\mathcal{R}$ can be defined by the line integral

$$(\mathcal{R}f)(r,\theta) = \int_{\mathcal{L}_{r,\theta}} f(x,y)\,\mathrm{d}\ell, \tag{1}$$

  where the line $\mathcal{L}_{r,\theta}$ is defined in figure 1. We refer to $\mathcal{R}f$ as the sinogram associated to the slice f. We note $p_\theta(r) = (\mathcal{R}f)(r,\theta)$ the *projection* of f obtained at angle θ.

- The Fourier slice theorem establishes a relationship between a projection and the corresponding slice in the Fourier domain

$$\hat{p}_\theta(\rho) = \hat{f}(\rho\cos\theta, \rho\sin\theta). \tag{2}$$

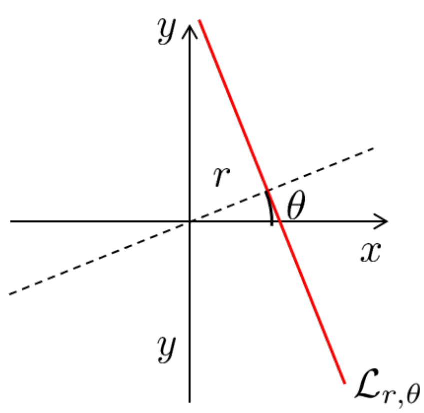


Figure 1: Coordinates used in the definition of the Radon transform.

- We finally define the back projection operator $\mathcal{B}$ by

$$(\mathcal{B}g)(x,y) = \int_0^\pi g(x\cos\theta + y\sin\theta, \theta)\,\mathrm{d}\theta, \tag{3}$$

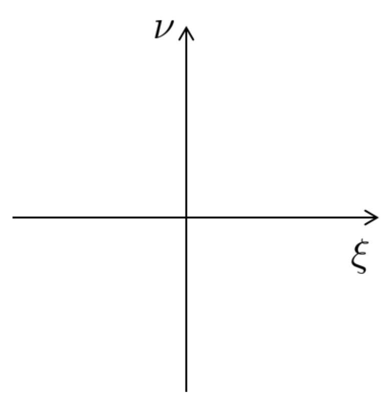
where $g(r,\theta)$ is typically a sinogram.

Part A. Fourier slice theorem

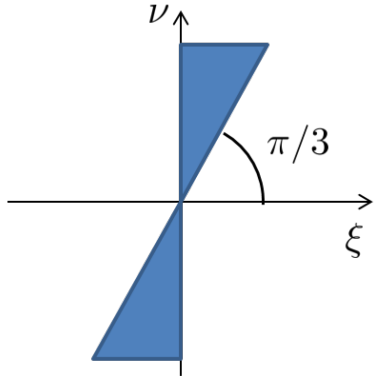
1. Define the two Fourier transforms involved in (2). Do they act on 1D, 2D, nD functions? Across which variables?

> **Solution:** The left-hand term is a 1D Fourier transform along the spatial (not angular) dimension $\hat{p}_\theta(\rho) = \int_{\mathbb{R}} p_\theta(r) \exp\left[-j2\pi\rho r\right]\,\mathrm{d}r$, while the right-hand term is a 2D Fourier transform $\hat{f}(\xi,\nu) = \iint_{\mathbb{R}^2} p_\theta(r) \exp\left[-j2\pi(\xi x + \nu y)\right]\,\mathrm{d}\xi\,\mathrm{d}\nu$

2. Draw the portion of the Fourier domain that is made available by the Fourier slice theorem assuming projections are measured for angles $\theta \in [\frac{\pi}{3}, \frac{\pi}{2}]$. Does such an acquisition allow to recover the slice $f(x,y)$ by inverse Fourier transform?

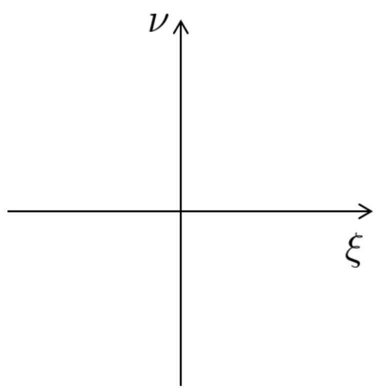


> **Solution:**

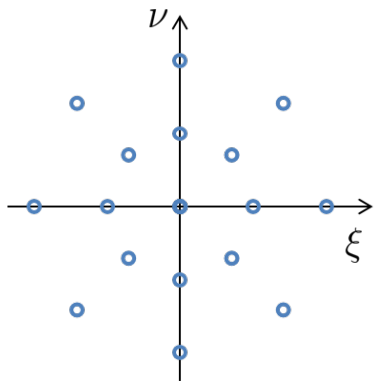


As most frequencies are missing, the slice will be blurred.

3. When projections are available at angles $\theta \in [0, \pi)$, does the Fourier slice theorem provide a practical reconstruction strategy? Illustrate your answer by drawing the portion of the Fourier domain that is measured for $\theta \in \{0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\}$ assuming the detector has 5 pixels.



Solution:



The sampling density is higher at lower frequencies. The signal has to be interpolated on a regular grid before inverse Fourier transformation. Here again the image will suffer from reconstruction artifacts.

Part B. Exact reconstruction formula

4. Give an integral expression of $f(x, y)$ as a function of its Fourier transform $\hat{f}(\xi, \nu)$

Solution: By inverse transformation, we have

$$f(x, y) = \iint_{\mathbb{R}^2} \hat{f}(\xi, \nu) \exp\left[j2\pi(\xi x + \nu y)\right] \,\mathrm{d}\xi \,\mathrm{d}\nu$$

5. What change of variable $(\xi, \nu) = \varphi(\rho, \theta)$ allows to exploit the Fourier slice theorem? How can it be interpreted?

Solution:

$$\begin{cases} \xi &= \rho \cos\theta \\ \nu &= \rho \sin\theta \end{cases}$$

This corresponds to a transformation from polar coordinates to Cartesian coordinates.

6. Compute the determinant of the Jacobian matrix of φ.

Solution: The Jacobian matrix is built from the first-order derivatives of φ. By definition

$$\begin{vmatrix} \frac{\partial \xi}{\partial \rho} & \frac{\partial \xi}{\partial \theta} \\ \frac{\partial \nu}{\partial \rho} & \frac{\partial \nu}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -\rho\sin\theta \\ \sin\theta & \rho\cos\theta \end{vmatrix} = \rho\cos^2\theta + \rho\sin^2\theta = \rho$$

7. Using the change of variable of Question 5, derive an expression of $f(x, y)$ as a function of $\hat{p}_\theta(\rho)$

Solution:

$$\begin{aligned} f(x,y) &= \iint_{\mathbb{R}^2} \hat{f}(\rho\cos\theta, \rho\sin\theta) \exp\left[j2\pi\rho(\cos\theta x + \sin\theta y)\right] \, \mathrm{d}\xi \, \mathrm{d}\nu \\ &= \iint_{\mathbb{R}^2} \hat{p}_\theta(\rho) \exp\left[j2\pi\rho(\cos\theta \cdot x + \sin\theta \cdot y)\right] \, \mathrm{d}\xi \, \mathrm{d}\nu \\ &= \int_0^\pi \int_{\mathbb{R}} \hat{p}_\theta(\rho) \exp\left[j2\pi\rho(\cos\theta \cdot x + \sin\theta \cdot y)\right] |\rho| \, \mathrm{d}\rho \, \mathrm{d}\theta \end{aligned}$$

8. It can be shown that $f = \mathcal{B}g$, where $\mathcal{B}$ is the back propagation operator defined by (3). Derive an expression for $g(r,\theta)$ as a function of $\hat{p}_\theta(\rho)$.

Solution: By identification from the previous answer, we have

$$g(r,\theta) = \int_{\mathbb{R}} \hat{p}_\theta(\rho) \exp\left(j2\pi\rho r\right) |\rho| \, \mathrm{d}\rho$$

9. Rewrite $g(r,\theta) = \mathcal{F}^{-1}\left[\hat{p}_\theta^{\text{filt}}\right](r)$ and give an expression of the filtered projections $\hat{p}_\theta^{\text{filt}}$. Note that $\mathcal{F}^{-1}$ is a 1D inverse Fourier transform along the spatial dimension.

Solution: By inverse Fourier transformation, we get

$$g(r,\theta) = \mathcal{F}^{-1}\left[\hat{p}_\theta^{\text{filt}}\right](r)$$

where we identify

$$\hat{p}_\theta^{\text{filt}}(\rho) = |\rho| \cdot \hat{p}_\theta(\rho)$$

Part C. Implementation

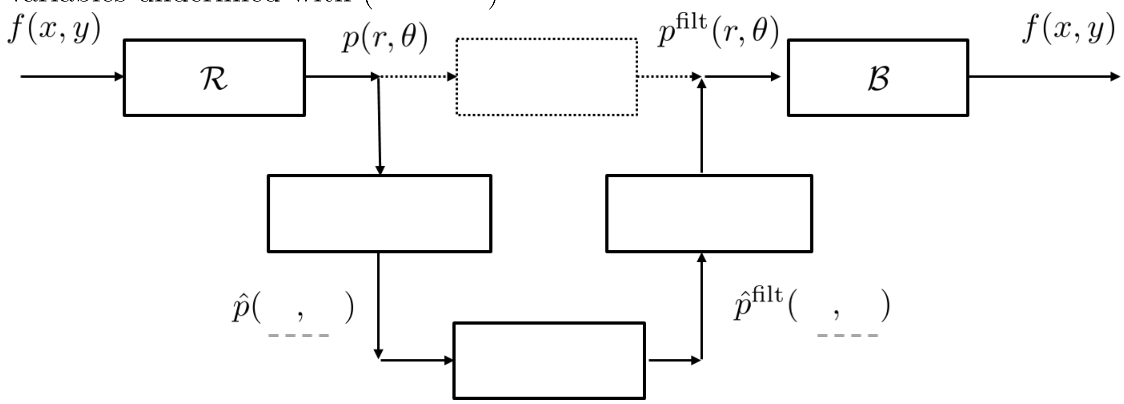
10. How to compute $g(r,\theta) = \mathcal{F}^{-1}\left[\hat{p}_\theta^{\text{filt}}\right](r)$ while avoiding direct and inverse Fourier transformations? We will note $\hat{h}(\rho) = |\rho|$ and use convolutions.

Solution: We simply notice that

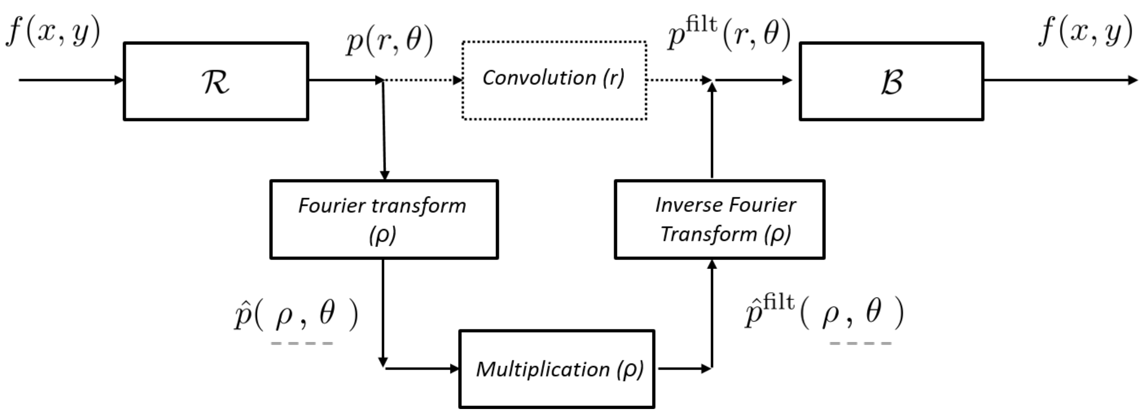
$$g(.,\theta) = \mathcal{F}^{-1}\left[\hat{p}_\theta^{\text{filt}}\right] = \mathcal{F}^{-1}\left[\hat{p}_\theta \cdot \hat{h}\right] = \mathcal{F}^{-1}\left[\hat{p}_\theta\right] * \mathcal{F}^{-1}\left[\hat{h}\right] = p_\theta * h$$

By filtering the sinogram along the spatial coordinate with the filter h Fourier transformations are no longer necessary.

11. Complete the diagram below. In each box, specify the operation and variables (e.g., Fourier transform (x, y), Fourier transform (r), convolution (θ), etc.). Specify the missing variables underlined with $(- - --)$



Solution:

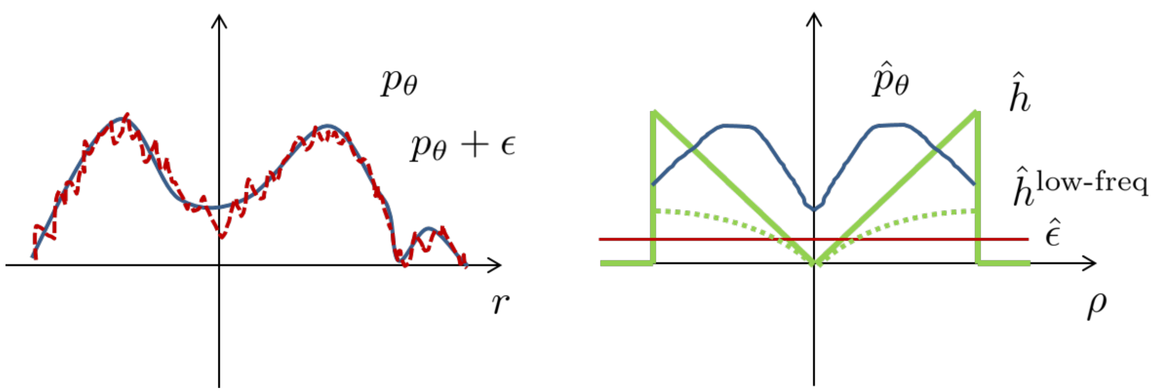


12. How do you categorize the filter $h(r)$? How to implement the filtering operation in practice?

Solution: The filter has an infinite impulse response (IIR), which requires an appropriate truncation to be implemented in practice. As the projections are discrete, their bandwidth B is limited by the Nyquist frequency. The same window B can be applied to the ramp filter in the Fourier domain. This has been discussed in detail by G. Zeng (IEEE Trans Nucl Sci., 62(1): 131–136, 2015).

13. Model the acquisition of noisy projections. How does the filtered back projection formula behave in the case of noise? Illustrate this behavior by drawing the profiles of the projections in the spatial and Fourier domains.

Solution: We measure $p_\theta(r) + \epsilon$. The ramp filter amplifies noise. To mitigate this problem, filters with lower gain at high frequencies are classically considered (e.g., Shepp-Logan, Cosine, Hamming).



Tdsi AR
26/01/2024
Durée : 15 minutes

NOM : ____________________
Prénom : ____________________

Cet énoncé est composé de 3 pages (y compris celle-ci). Merci de compléter vos nom et prénom en haut à droite de la première page et de placer vos initiales sur les pages suivantes.

Seules vos notes de cours sont autorisées.

QCM (4 points)

N.B. : **Une seule** bonne réponse est possible.

1. Quelle résolution spatiale peut-on atteindre avec une sonde de 5 MHz dans les tissus biologiques ? Rappel : la vitesse de propagation des ultrasons dans l'eau est de 1540 m/s.
   - ◯ Environ 3 cm.
   - ◯ Environ 300 μm.
   - ◯ Largement moins de 300 μm.
   - ◯ Bien plus de 3 cm.

2. On souhaite focaliser des ondes ultrasonores au point $(x, z) = (0, 2)$ cm (cf. Figure 1). Quel retard appliquer à l'élément situé en $x = 0$ pour que l'onde émise par cet élément arrive en même temps que l'onde émise par un élément situé en $x = 600$ μm ? On supposera une vitesse de propagation des ultrasons de 1540 m/s.
   - ◯ Aucun retard n'est nécessaire.
   - ◯ 6 ns.
   - ◯ 13 μs.
   - ◯ 26 μs.

3. Une onde transverse
   - ◯ implique un mouvement local du milieu qui est colinéaire à la direction de propagation de l'onde.
   - ◯ n'implique aucun mouvement local du milieu.
   - ◯ implique un mouvement local du milieu qui est perpendiculaire à la direction de propagation de l'onde.
   - ◯ entraine une confusion globale de la propagation.

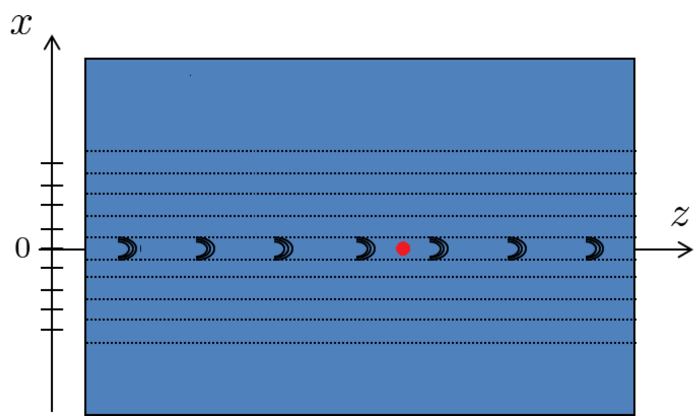


FIGURE 1 – Géométrie du problème de la question 3.

4. Une onde ultrasonore traverse un milieu d'atténuation 1 dB/cm/MHz et d'épaisseur 4 cm. Quelle sera l'atténuation de l'onde en sortie ?
   - ◯ On ne peut pas savoir.
   - ◯ 8 dB.
   - ◯ 4 dB.
   - ◯ 20 dB.

5. La tomographie par rayons X consiste à reconstruire l'image de l' . . . . . . d'un objet à partir d'un ensemble de . . . . . . acquises pour . . . . . .
   - ◯ amplitude / prédictions / pour le même angle de vue.
   - ◯ amplitude / projections / pour 180 angles de vue.
   - ◯ atténuation / prédictions / pour 360 angles de vue.
   - ◯ atténuation / projections / différents angles de vue bien choisis.
   - ◯ atténuation / prédictions / pour 180 angles de vue.
   - ◯ amplitude / projections / différents angles de vue bien choisis.

6. Le signal mesuré par un détecteur de rayons X est proportionnel
   - ◯ au nombre de photons incidents sur l'objet.
   - ◯ à l'atténuation de l'objet.
   - ◯ à la texture et la forme de l'objet.
   - ◯ l'angle de vue optimal.

7. Comment réduire le temps d'une acquisition tomographique ?
   - ◯ En augmentant le nombre d'angles de vue acquis.
   - ◯ En diminuant le nombre de pixels du détecteur.
   - ◯ En augmentant le nombre de pixels du détecteur.
   - ◯ En diminuant le nombre d'angles de vue acquis.
   - ◯ En utilisant un algorithme de reconstruction rapide.